Mathematical Reviews

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ALGEBRA

***Albert, A. Adrian. Introduction to Algebraic Theories.** University of Chicago Press, Chicago, 1941. viii+137 pp. $1.75.

There is a gap in mode of thought between the usual intuitive first course in the theory of equations and the rigorous abstract treatment of modern higher algebra. To make an appropriate transition from the intuitive to the abstract, Albert proposes adding a new course in algebra to the undergraduate curriculum in mathematics. The book is a text for such a course. It opens with a chapter on polynomials. Then rectangular matrices are treated; the equivalence of matrices and bilinear forms is studied. The fourth chapter deals with linear spaces and linear equations. It is followed by a chapter on polynomials with matric coefficients, elementary divisors, similarity of matrices. The concluding chapter introduces the reader to the fundamental concepts of modern algebra. Groups, rings and fields are defined; the notions of ideals and residue class rings are discussed. The arithmetic in the ring of Gauss integers is developed, and an example of a quadratic field is given in which not every ideal is a principal ideal. In this short review it is not possible to mention all the places in the book where the author gives a new "turn" to known theories. The exposition is careful and clear, and a large number of exercises illustrate the theory. The book forms a very good introduction to Albert's Modern Higher Algebra.

R. Brauer (Toronto, Ont.).

***MacDuffee, Cyrus Colton. An Introduction to Abstract Algebra.** John Wiley & Sons, Inc., New York, 1940. vii+303 pp. $4.00.

The purpose of this book is to serve as a text for a first graduate course in modern algebra. An outline of the contents according to chapters is as follows: I. Theory of numbers. Peano's axioms. Unique factorization. Linear congruences. Fermat's theorem. Polynomial congruences. Primitive roots. Quadratic residues and the quadratic reciprocity law. II. Finite groups. Elementary concepts and properties. Order of a subgroup. Quotient groups. Jordan-Hölder theorem. Direct products. Permutation groups. Basic theorem for Abelian groups. III. Algebraic fields. Deals mainly with subfields of the complex field. Stem field and root field of a polynomial. Automorphisms. Galois group. Roots of unity. Solvability of equations by radicals. IV. Integral algebraic domains. Example of quadratic domains treated first. Ideals. Unique factorization in maximal integral domains of algebraic number fields. Discriminant and norm. V. Rings and fields. Elementary properties of abstract algebraic systems. Quotient field. Characteristic of a field. Polynomial rings. Difference rings. Homomorphism. Algebraic and transcendental extension of fields. Theory of (finite) Galois fields. VI. Perfect fields. Construction of field of real numbers. Algebraic closure of complex field. Construction of field of p-adic numbers. Indication of extension to algebraic number fields. VII. Matrices. Vectors and linear systems. Linear independence. Determinant criteria for linear independence. Linear equations. Algebra of matrices. Hamilton-Cayley theorem and theory of minimum polynomial. Equivalence of matrices with elements in a principal ideal ring and application to similarity problem for matrices over a field. VIII. Linear associative algebras. Example of quaternion algebra. Arithmetic theory of quaternions. Generalized quaternions and cyclic algebras. Definition of algebra with a finite basis. Regular representations. Direct product and direct sum. The radical.

The book is very readable and the wealth of concrete examples should make it a useful text. The following errors were noted: p. 108. The theory of cyclotomic fields given here applies only to the case where $\mathfrak{F}$ is the field of rational numbers. p. 111. The proof of the necessity of Theorem 47.3 is incomplete. p. 175. The proof that $GF(p, m(\lambda))$ depends only on the degree of $m(\lambda)$ tacitly assumes that the root field is unique. No proof is given of this important theorem. The reviewer believes that the author's use of the term algebraic variety for algebraic system is unfortunate, since the former has a definite significance in algebraic geometry. *N. Jacobson* (Baltimore, Md.).

Equations, Polynomials

Kantz, Georg. Neue Herleitung der Darstellung der Potenzsummen der Wurzeln eines normierten Polynoms n-ten Grades von x durch seine Koeffizienten. Deutsche Math. 5, 393–394 (1941). [MF 4250]

An elementary proof of Newton's formulas, somewhat simpler than that found in such elementary texts as Chrystal, Algebra, 1886, part 1, p. 423. *P. Franklin.*

Madhava Rao, B. S. and Sastry, B. S. On the limits for the roots of a polynomial equation. J. Mysore Univ. Sect. B. 1, 5–8 (1940). [MF 3622]

A simple proof is here given for Laguerre's theorem that, if the zeros of a polynomial

$$f(x)=x^n+nax^{n-1}+n(n-1)bx^{n-2}+n(n-1)(n-2)cx^{n-3}+\cdots$$

are all real and distinct, then the zeros lie between the limits $-a\pm(n-1)(a^2-2b)^{\frac{1}{2}}$; that is, between the zeros of the polynomial

$$\psi(x)=(n-1)(s_2-x^2)-(x-s_1)^2,$$

where s_1 is the sum of the zeros $\alpha, \beta_1, \beta_2, \cdots, \beta_{n-1}$ of $f(z)$ and s_2 is the sum of their squares. In the present paper, the theorem is verified by showing that

$$0<\sum(\beta_i-\beta_j)^2\equiv\psi(\alpha).$$

The authors also consider the relation

$$0<\sum(\beta_i-\beta_j)^4\equiv\varphi(\alpha).$$

The fourth degree polynomial $\varphi(x)$, whose coefficients depend only upon a, b, c and n, has only two real zeros L_1 and L_2. These zeros provide better (but more complicated) limits than the above for the zeros of $f(x)$. Besides referring to Laguerre's proof [p. 92, vol. I of his Collected Works], his very elementary proof given in the footnote on p. 93 [ibid.] might have been mentioned. *M. Marden.*

Weisner, Louis. Moduli of the roots of polynomials and power series. Amer. Math. Monthly **48**, 33–36 (1941). [MF 3721]

The paper contains theorems connected with Cauchy's theorem: In an equation $f(z)=c_0+c_1z+\cdots+c_nz^n$, $c_n\neq 0$, c_j complex, the absolute value of every root is not greater than the sole positive root of

$$|c_0|+|c_1|z+\cdots+|c_{n-1}|z^{n-1}-|c_n|z^n=0,$$

and with Pellet's theorem: If

$$|c_0|+\cdots+|c_{\nu-1}|z^{\nu-1}-|c_\nu|z^\nu+|c_{\nu+1}|z^{\nu+1}+\cdots+|c_n|z^n=0, \quad 0<\nu<n$$

has two positive roots r_1, r_2 ($r_2\geqq r_1$), then $f(z)=0$ has exactly ν roots of absolute value not greater than r_1. There are references to results by Walsh, Egervary, Pólya-Szegö, Kakeya, Anghelutza, Montel, Chang.

The following theorems are established. Assume the equation

$$f_m(z)=a_0z^{n_0}+\cdots+a_mz^{n_m}, \quad 0\leqq n_0<n_1<\cdots<n_m;\ a_s\neq 0;\ m\geqq 1.$$

Let $q_s=n_s-n_{s-1}$. Theorem: For k fixed, $0\leqq k\leqq m$, let $p_1,\cdots,p_m$ be positive numbers, $p_0=1$, and $p_0+p_0p_1+p_0p_1p_2+\cdots+p_0p_1p_2\cdots p_m\leqq 2p_0p_1\cdots p_k$, and let $u_s=|(p_sa_{s-1})/a_s|^{1/q_s}$, $s=1,\cdots,m$. If $R\geqq u_s$, $s=1,\cdots,k$, $R\leqq u_s$, $s=k+1,\cdots,m$, the circle $|z|\leqq R$ contains exactly n_k roots of $f_m(z)=0$. Special cases: If $R\geqq|(3a_{s-1})/a_s|^{1/q_s}$, $s=1,\cdots,k$, $R\leqq|a_{s-1}/(3a_s)|^{1/q_s}$, $s=k+1,\cdots,m$, the circle $|z|\leqq R$ contains exactly n_k roots. In case $k=m$, the absolute value of every root of $f_m(z)=0$ is not greater than the largest of the numbers $|(\tau_ma_{s-1})/a_s|^{1/q_s}$, $s=1,\cdots,m$, τ_m the positive root of $1+z+\cdots+z^{m-1}-z^m=0$; and, still in case $k=m$, the absolute value of every root is not greater than the largest of $|a_0/a_1|^{1/q_1}$, $|(2a_{s-1})/a_s|^{1/q_s}$, $s=2,\cdots,m$. Two applications to roots of partial sums of power-series are given. *A. J. Kempner* (Boulder, Colo.).

Erdös, P. Note on some elementary properties of polynomials. Bull. Amer. Math. Soc. **46**, 954–958 (1940). [MF 3455]

In a previous paper [Ann. of Math. (2) **40**, 537–548 (1939); these Rev. **1**, 1] Erdös and T. Grünwald proved the following theorem: If $f(x)$ is a polynomial of degree not less than 2, with only real roots, $f(-1)=f(1)=0$, $f(x)>0$ $(-1<x<1)$, $\max_{-1<x<1}f(x)=1$, then $\int_{-1}^{1}f(x)dx\leqq 4/3$, with equality occurring only if $f(x)=1-x^2$. This extremal property of the parabola is now extended as follows. Let $0<f(a)=f(b)=d<1$, $-1<a<b<1$, then $b-a\leqq 2(1-d)^{\frac{1}{2}}$ with equality occurring again only if $f(x)=1-x^2$. This means that the span $b-a$ of a horizontal section, at fixed level $y=d$, of the graph of $y=f(x)$ $(-1\leqq x\leqq 1)$ reaches its maximum value $2(1-d)^{\frac{1}{2}}$ only for the parabola $y=1-x^2$. A number of suggestive related extremal properties of polynomials are conjectured at the end of the note.

I. J. Schoenberg (Waterville, Me.).

Witt, Ernst. Ein Identitätssatz für Polynome. Mitt. Math. Ges. Hamburg **8**, part 2, 188–189 (1940). [MF 4121]

Let

$$f_n(z)=\prod_{i=1}^{m}(1-a_i^nz) \quad\text{and}\quad g_n(z)=\prod_{i=1}^{m}(1-b_i^nz),$$

where $|a_i|<1$ and $|b_i|<1$. A new proof is given that, if $f_n(1)=g_n(1)$ for $n=1,2,\cdots$, then $f_1(z)\equiv g_1(z)$.

A. C. Schaeffer (Stanford University, Calif.).

Thomas, J. M. The resolvents of a polynomial. Amer. Math. Monthly **47**, 686–694 (1940). [MF 3646]

Let $f(z)$ be a polynomial of degree n with complex coefficients. Write $z=x+iy$ and $f(x+iy)=g(x,y)+iyh(x,y)$, where the terms of odd degree have been segregated so that g, h are polynomials in x, y^2. Denote by R_x the resultant of g, h written as polynomials in y^2, and by R_y the resultant of g, h written as polynomials in x. The roots of R_x are the $\frac{1}{2}n(n-1)$ arithmetic means of the roots of f taken in pairs, and if the roots of R_y be multiplied by -4, they become the $\frac{1}{2}n(n-1)$ squares of the differences of the roots of f. It is shown that either R_x or R_y may be used to prove the fundamental theorem of algebra by induction on the degree. The method is practicable to the extent that it may be used to solve the quartic equation.

C. C. MacDuffee (New York, N. Y.).

Geronimus, J. The generalization of a lemma of M. S. Kakeya. Bull. Amer. Math. Soc. **47**, 93–95 (1941). [MF 3817]

Generalization of a result of Kakeya concerning the explicit representation of certain types of polynomials in the complex domain. *J. A. Shohat* (Philadelphia, Pa.).

Engstrom, H. T. Polynomial substitutions. Amer. J. Math. **63**, 249–255 (1941). [MF 4147]

Let $F(x)$ be a polynomial in x, of degree greater than unity, with coefficients in a field of characteristic zero. $F(x)$ is called reducible or irreducible according as there do or do not exist two polynomials $f(x)$ and $g(x)$, of degrees greater than unity and with coefficients in the given field, such that $F(x)=f(g(x))$. It is obvious that every $F(x)$ can be obtained by compounding a finite number of irreducible polynomials. The author proves that two decompositions of $F(x)$ into irreducible polynomials contain the same number of irreducible polynomials, the degrees of the irreducible polynomials in the two decompositions being the same, except, perhaps, for the order in which they occur. The proof is purely algebraic. For the field of complex numbers this question had been treated by J. F. Ritt, who used methods of the theory of functions. *J. F. Ritt* (New York, N. Y.).

Brewer, B. W. A criterion for solvability by radicals. Amer. J. Math. **63**, 119–126 (1941). [MF 3635]

A polynomial $f(x)$ in a field K is said to be solvable by radicals over K if there exists a chain of fields $K_0=K\subset K_1\subset\cdots\subset K_s\supseteqq W_f$, where W_f is the root field of $f(x)$ over K, and $K_i=K_{i-1}(\alpha_i)$, where α_i is a root of an irreducible binomial of prime degree in K_{i-1}. Let n be the order of the Galois group G of $f(x)$ relative to K. The author proves that $f(x)$ is solvable by radicals over K if and only if (I) G is solvable, and (II) primitive nth roots of unity exist over K and the cyclotomic polynomial in K whose roots are the $\varphi(n)$ distinct primitive nth roots of unity is solvable by radicals over K. When the characteristic of K is 0 this reduces to the Galois criterion (I), that is, that the com-

position factors of G be primes. In case the characteristic of K is a prime p, the author obtains a similar criterion in terms of the composition factors of G, namely, that these be primes of a certain class defined by p and the Steinitz degree m of the maximal, absolutely algebraic subfield of K.

R. Hull (Vancouver, B. C.).

Linear Algebra

Tricomi, Francesco. Sul teorema di Hadamard sui determinanti. Univ. Nac. Tucumán. Revista A. **1**, 297–301 (1940). [MF 4069]

The author gives a new proof of Hadamard's theorem on the maximum value of a determinant whose elements are restricted in absolute value. The essential part of the proof is the following: If A is a real square matrix of order n, there exists a triangle matrix T with units in the main diagonal such that $B=TA$ and BB' is a diagonal matrix. Then, with the usual notation,

$$\sum_{i=1}^{n} b_{ri}^2 \leqq \sum_{i=1}^{n} a_{ri}^2.$$

J. Williamson (Baltimore, Md.).

Sagastume Berra, Alberto E. Determinants and linear equations in quasi-fields. Univ. Nac. Tucumán. Revista A. **1**, 123–141 (1940). (Spanish) [MF 4059]

Starting with a matrix of order n with elements in a quasi-field Q, the author defines $(n!)^2$ different determinants of order n, each of these being of total degree $2n-2$ in the elements of the given matrix. While these determinants will, in general, have different values, it is true that if one vanishes they all vanish. After establishing a number of properties of these determinants, the author passes to the solution of systems of linear equations. The principal result is an extension of Cramer's rule for obtaining the solution of a system of equations $\sum_{j=1}^{n} x_j a_{ij} = y_i$ $(i=1, 2, \cdots, n)$, where the coefficients are in Q, the determinants of the coefficients being different from zero. The work is rather closely related to that of Richardson [Proc. London Math. Soc. **28**, 395–420 (1928)] and Ore [Ann. of Math. (2) **32**, 463–477 (1931)]. *N. H. McCoy* (Northampton, Mass.).

Albert, A. A. A rule for computing the inverse of a matrix. Amer. Math. Monthly **48**, 198–199 (1941). [MF 4135]

Let A be a given numerical matrix of order n, and I the unit matrix of the same order. Apply elementary row transformations to A which carry it into I. Then A^{-1} can be obtained by applying these same transformations to I.

N. H. McCoy (Northampton, Mass.).

Scott, W. M. On characteristic roots of matrix products. Amer. Math. Monthly **48**, 201–203 (1941). [MF 4137]

A new proof of the known theorem that, if C_1 and C_2 are square matrices of the same order, then C_1C_2 and C_2C_1 have the same characteristic roots. *N. H. McCoy*.

Ingraham, Mark H. and Trimble, H. C. On the matric equation $TA=BT+C$. Amer. J. Math. **63**, 9–28 (1941). [MF 3625]

The solution of this matrix equation for matrices with elements in a field or a division algebra is reduced to a problem in polynomial congruences. In the commutative case the latter may be solved constructively. It is shown that the maximum rank of T such that $TA=BT$ is the sum of the degrees of the common divisors of corresponding invariant factors of A and B. To study the matrices commutative with A the authors use a known correspondence between such matrices and certain matrices with polynomial elements. By this method they prove that if A has k invariant factors and B commutes with A then equations of the form $h_1(A)=0$, $g_1(A)[B-a_{10}(A)]=0, \cdots$, $B^r-a_{r,r-1}(A)B^{r-1}-\cdots-a_{r0}(A)=0$ hold for $r\leqq k$. For nonderogatory A this gives the known theorem that B is a polynomial in A. The extension of these results to the division algebra case is indicated. *N. Jacobson*.

Ingraham, Mark H. Rational methods in matrix equations. Bull. Amer. Math. Soc. **47**, 61–70 (1941). [MF 3811]

This paper is concerned with the unilateral matrix equation $\sum A_i X^i = 0$, where the A_i are given $n\times n$ matrices over a field. After an exposition of known results, the author gives a new algorithm for the solution of this equation.

N. H. McCoy (Northampton, Mass.).

Taussky, Olga and Todd, John. Matrices of finite period. Proc. Roy. Irish Acad. Sect. A. **46**, 113–121 (1941). [MF 3864]

It is proved that for a fixed modulus m every two-by-two matrix with integral elements and determinant unity is congruent, modulo m, to a matrix of finite period if and only if $m=2$ or 3. To prove that this can be done for $m=2$, 3 the authors construct a complete set of matrices, modulo m $(m=2, 3)$, each of which is of finite order. Since it is immediately evident that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not congruent to a matrix of finite period of $m>3$, the theorem follows. The authors also discuss the matrices whose elements are complex (Gaussian) integers and find in the same way that the modulus $m=1+i$ is the only one for which every such matrix is congruent to one of finite period.

H. W. Brinkmann (Swarthmore, Pa.).

Comét, Stig. Über Sondermatrizen und ihre Verwendung. Proc. [Förhandlingar] Roy. Physiographic Soc. Lund **10**, 77–97 (1941). [MF 4288]

A sequence $[A]$ of matrices $A^0, A^1, A^2, \cdots$ (whose superscripts need not be exponents) is recurrent in case a minimum function (Grundcharakteristik) $f(x)$ exists such that $A^j f(A)=0$ for all $j\geqq 0$. If

$$f(x)=(x-x_1)^{\nu_1}(x-x_2)^{\nu_2}\cdots(x-x_r)^{\nu_r},$$

where the x_i are different and each $\nu_i\geqq 1$, the general term A^j can be uniquely written

$$A^j=M_1{}^j+M_2{}^j+\cdots+M_r{}^j,$$

where $[M_i]$ has the minimum function $(x-x_i)^{\nu_i}$ and $M_i{}^j$ is expressible linearly in terms of the A^j. If $[A]$ is a power series, so are the $[M_i]$, and the terms of $[M_i]$ are orthogonal to those of $[M_j]$, $i\neq j$. A matrix S is called special (Sondermatrix) of index (Verknüpfungsgrad) ν if $S(S-E)^\nu=0$, where ν is minimal. Thus a special matrix of index 1 is idempotent. If S^0 is by definition $E-(E-S)^\nu$, then a necessary and sufficient condition that a matrix have the minimum function $(1-x)^\nu$ is that it be special of index ν. For every integer $j\geqq 0$,

$$S^j=S^0+Kj/1+K^2j^2/2!+\cdots+K^{\nu-1}j^{\nu-1}/(\nu-1)!,$$

where K is a nilpotent matrix of index ν such that $KS^0=S^0K=K$. Conversely every idempotent S^0 and nilpotent K of index ν such that $KS^0=S^0K=K$ define a special matrix S by means of the above formula. For j fractional, this formula defines the principal value of a root of S. The complete solution in matrices with complex elements of the scalar matric equation $f(X)=0$ can be given in terms of these special matrices. If

$$f(x)=x^{\nu_0}(x-x_1)^{\nu_1}\cdots(x-x_s)^{\nu_s},$$

where $x_i\neq 0$, $x_i\neq x_j$, then

$$X=M_0+x_1S_1+\cdots+x_sS_s,$$

where $M_0, S_1, \cdots, S_s$ are orthogonal, M_0 is 0 or nilpotent of index not greater than ν_0, and each S_i is 0, or special of index not greater than ν_i. *C. C. MacDuffee* (New York, N. Y.).

van Kampen, E. R. Elementary proof of a theorem on Lorentz matrices. Bull. Amer. Math. Soc. **47**, 288–290 (1941). [MF 4178]

The author sketches a topological proof and gives an algebraic proof of the theorem that the signs of the determinant $|A|$ and $|D|$ form two 1-dimensional representations of the group of Lorentz matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of $(n+m)$ dimensional real linear transformations leaving invariant the form x^2-y^2, where x and y are n- and m-vectors. This proof, which is valid for any real field, is more elementary than other proofs which have been given [cf. W. Givens, Bull. Amer. Math. Soc. **46**, 81–85 (1940); these Rev. **1**, 195]. *B. W. Jones* (Ithaca, N. Y.).

Werjbitzky, B. D. On the summation of series of products of several matrices. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 100–110 (1940). (Russian) [MF 3305]

In this paper the author considers an infinite series

$$(1)\quad F(X_1, \cdots, X_m)=\alpha_0+\sum_{\nu=1}^{\infty}\sum_{j_1,\cdots,j_\nu}^{(1,2,\cdots,m)} X_{j_1}X_{j_2}\cdots X_{j_\nu}\alpha_{j_1j_2\cdots j_\nu}$$

with scalar coefficients $\alpha_0, \alpha_{j_1\cdots j_\nu}$, in m matrices $X_1, \cdots, X_m$ of order n, not all of which are commutative with one another. For $n=2$ he shows that (1) may be written as a quadratic polynomial $M_0'+M_1'X_p+M_2'X_q+M_3'X_pX_q$ in any pair of non-commutative matrices X_p, X_q among X_i. The scalar coefficients M_i are series in rational functions of the traces $\sigma(X_i)$, $\sigma(X_pX_j)$, $\sigma(X_qX_j)$, $\sigma(X_pX_qX_j)$. Since, also, M_i' may be expressed rationally in terms of $\sigma(X_p)$, $\sigma(X_q)$, $\sigma(X_pX_q), \cdots, \sigma((X_pX_q)^2)$ and in terms of (2) $\sigma(F)$, $\sigma(X_pF)$, $\sigma(X_pX_qF)$, it can be seen that (1) converges if and only if the scalar series (2) converge simultaneously. In the conclusion of the paper the author points out that by considerations similar to those in the case $n=2$ one can prove that, for any n, any analytic function (1) may be expressed as a polynomial

$$R_1X_p^{n-1}X_q^{n-1}+R_2X_p^{n-1}X_q^{n-2}+\cdots+R_{n^2},$$

where the scalar coefficients R_i are series in rational functions of $n^2m-(n^2-1)$ traces of the matrices $X_p^rX_q^s$, $X_p^nX_q^n$, $X_p^rX_q^sX_j$ ($r, s=0, 1, \cdots, n-1$; $j=1, \cdots, m$; $j\neq p$ or q). *A. E. Ross* (St. Louis, Mo.).

Jacobsthal, Ernst. Zur Hauptachsentransformation einer positiv definiten quadratischen Form. Norske Vid. Selsk. Forh. **13**, 119–122 (1940). [MF 3871]

Satterthwaite, Franklin E. A concise analysis of certain algebraic forms. Ann. Math. Statistics **12**, 77–83 (1941). [MF 4004]

It is shown how the use of the Kronecker delta and the summation convention of the tensor calculus can make the writing of certain linear and quadratic forms commonly occurring in statistics and calculations with them quite concise. This notation is also used in the calculation of the ranks of matrices of the form $C\equiv(\alpha_{ij}\beta_{kl})$ which also occur often in statistics, and which are called the uncontracted product of $A\equiv(\alpha_{ij})$ and $B\equiv(\beta_{kl})$. It is proved that the rank of C is the product of the ranks of A and B. The paper contains several illustrative examples. *C. C. Craig.*

Oldenburger, Rufus. The minimal number problem for binary forms. Proc. Nat. Acad. Sci. U. S. A. **27**, 185–188 (1941). [MF 3902]

Let F be a binary form of degree n with coefficients in a field K which is mildly restricted. The least number m of nth powers of linear forms as a linear combination of which F can be expressed is called the minimal number of F for K. The author forms rectangular matrices $A_0, \cdots, A_n$ from the coefficients of F such that each A_σ determines a covariant F_σ of degree σ. The smallest number γ such that $F_\gamma\neq 0$ is the "minimum associate degree" of F. If K is algebraically closed, either $m=\gamma$ or $m=n-\gamma+2$, while for fields in general either $m=\gamma$ or $m\geqq n-\gamma+2$.
C. C. MacDuffee (New York, N. Y.).

Abstract Algebra

Whitman, Philip M. Free lattices. Ann. of Math. (2) **42**, 325–330 (1941). [MF 3688]

The author solves the problem of describing explicitly the free lattice with $n\geqq 3$ generators X_i. He proves that (1) $X_i\leqq X_j$ if and only if $i=j$; (2) recursively, $A\leqq B$ if and only if $A=A'\cup A''$, where $A'\leqq B$ and $A''\leqq B$; or $A=A'\cap A''$, where $A'\leqq B$ or $A''\leqq B$; or $B=B'\cup B''$, where $A\leqq B'$ or $A\leqq B''$; or $B=B'\cap B''$, where $A\leqq B'$ and $A\leqq B''$. He further shows that, among all lattice polynomials representing the same element, those of shortest length are equivalent under the commutative and associative laws. Covering relations are also proved. Nowhere is n assumed to be finite or countable. *G. Birkhoff.*

Stabler, E. R. Sets of postulates for Boolean rings. Amer. Math. Monthly **48**, 20–28 (1941). [MF 3719]

A Boolean ring, as defined by Stone, is a ring in which every element is idempotent. The author's principal results are two sets of postulates for Boolean rings, sets (A) and (B) below. Set (A) is a weakened form of a set of sufficient conditions for Boolean rings given by Stone, and is as follows. (A): (1) If a, b are in the class, then $a+b$ and ab are in the class; (2) $(a+b)+c=a+(b+c)$; (3) if $a+b_1=a+b_2$, then $b_1=b_2$; (4) if $a_1+b=a_2+b$, then $a_1=a_2$; (5) $(ab)c=a(bc)$; (6) $a(b+c)=ab+ac$; (7) $(a+b)c=ac+bc$; (8) $aa=a$. Set (B) is as follows. (B): (1) If a, b are in the class then $a+b$ and ab are in the class; (2) $a+(b+c)=b+(c+a)$; (3) $(a+a)+b=b$; (4) $a(bc)=b(ca)$; (5) $a(b+c)=ab+ac$; (6) $aa=a$. Postulates

(B) the author found embedded in a set of postulates for Boolean algebra due to the reviewer. In both (A) and (B) a postulate for non-emptiness of the class is to be understood. The author also gives two slight variations of set (A), discusses duality with regard to (A), and raises the question of independence of the postulates of (A) when a postulate introducing a unit element is added. Independence is shown for (A) and (B), but not for the variations.

The author says of (A) and (B) that they contain no existence postulates, aside from the implied non-emptiness postulate. The term "existence" should be replaced by "unconditioned existence." All closure postulates are conditioned existence postulates. *B. A. Bernstein.*

McCoy, Neal H. Divisors of zero in matric rings. Bull. Amer. Math. Soc. **47**, 166–172 (1941). [MF 3836]

Let R be a commutative ring with unit element, R_n the ring of nth order matrices with elements in R, $R[\lambda]$ the polynomial ring with coefficients in R, and $R[A]$ the subring of R_n composed of polynomials in a matrix A of R_n. It is shown that A is a divisor of zero in $R[A]$ if and only if $|A|$ is a divisor of zero in R. The resultant $\mathcal{R}(f, g)$ of two polynomials $f(\lambda)$ and $g(\lambda)$ of $R[\lambda]$ is defined by the usual Sylvester determinant. If $f(\lambda)$ has leading coefficient unity, $g(\lambda)$ is prime to $f(\lambda)$ if and only if $\mathcal{R}(f, g)$ is not a divisor of zero in R. Let $h_{ij}(\lambda)$ denote the minors of $\lambda - A$ of order $n-1$. If every ideal in R has a finite basis, the matrix A is not derogatory if and only if the ideal $(h_{11}(\lambda), h_{12}(\lambda), \cdots, h_{nn}(\lambda))$ contains an element of R which is not a divisor of zero. *C. C. MacDuffee* (New York, N. Y.).

Murdoch, David C. and Ore, Oystein. On generalized rings. Amer. J. Math. **63**, 73–86 (1941). [MF 3631]

The axiomatic investigations of types of generalized groups [e.g., Hausmann and Ore, Amer. J. Math. **59**, 983–1003 (1937)] suggest a similar study of rings. In the case of groups the objective is the statement of various minimal sets of postulates sufficient to give the standard decomposition theorems. Similarly, the present paper proposes weakened axioms for the multiplication in a ring which will insure the validity of typical properties of ideal decomposition, multiplication and residuation.

A generalized ring R is an algebraic system with two binary operations of addition and multiplication which is a (not necessarily abelian) group under addition. A left ideal in R is a normal subgroup A of the additive group which is closed under multiplication by elements of R on the left. The "first distributive law" is the requirement that any expression $c(a+b)$ can be written in the form a_1+b_1, where a_1 and b_1 belong, respectively, to the left ideals generated by a and b. When this law holds the union of two left ideals A and B is the set of all sums $a+b$, for $a\epsilon A$, $b\epsilon B$, and furthermore the left ideals form a modular lattice. Sharper distributive laws must be assumed to obtain isomorphism theorems or to make the quotient system R/A a generalized ring. The product of two ideals A and B is the left ideal generated by all products of elements $a \cdot b$. Conditions are stated sufficient to make this product agree with the set of all sums $\sum a_i b_i$. One may obtain the important relation $(A \vee B)C = AC \vee BC$ if one assumes a certain generalized distributive law. This law is also sufficient to give the refinement theorem for a direct decomposition of R as a union of two-sided ideals. Left and right residuals may be defined after the manner of ideal quotients. The left residual is a left ideal if three appropriate axioms are assumed. Various other properties of residuals are systematically listed. The generalized rings include Lie rings. It is interesting that they also include groups: write the group operation as addition, and let the product be the commutator. The ideals are then exactly the normal subgroups, and the preceding theory applies. In particular, residuals (left=right) and "products" of normal subgroups may be introduced.

S. MacLane (Cambridge, Mass.).

Nakayama, Tadasi. Note on uni-serial and generalized uni-serial rings. Proc. Imp. Acad. Tokyo **16**, 285–289 (1940). [MF 2946]

The author adds some remarks to his papers "On Frobeniusean algebras" [I, Ann. of Math. (2) **40**, 611–633 (1939); these Rev. **1**, 3; II, Ann. of Math. (2) **42**, 1–21 (1941)]. He considers a ring A satisfying the minimum and maximum condition for left and right ideals and possessing a 1-element. He then proves that, if a two-sided ideal $\mathfrak{z}$ is expressible in the form $\mathfrak{z}=Ac=dA$ (c and d in A), then $\mathfrak{z}=cA=Ad$. Further, if every residue class ring of A is Frobeniusean, then A is uni-serial, and conversely. On account of these results, the author's characterization of Frobeniusean rings is equivalent to a characterization of uni-serial rings given by K. Asano [Jap. J. Math. **15**, 231–253 (1939); these Rev. **1**, 103]. Finally, a condition given previously by the author for generalized uni-serial algebras is extended to the case of generalized uniserial rings.

R. Brauer (Toronto, Ont.).

Nakayama, Tadasi. A correction to "A remark on the sum and the intersection of two normal ideals in an algebra." Bull. Amer. Math. Soc. **47**, 332 (1941). [MF 4187]

The paper in question appeared in the same Bull. **46**, 469–472 (1940); cf. these Rev. **1**, 327.

Matusita, Kameo. Ueber die Idealtheorie im Integritätsbereich mit dem eingeschränkten Vielfachenkettensatz. Proc. Phys.-Math. Soc. Japan (3) **23**, 8–12 (1941). [MF 3908]

Let $\mathfrak{J}$ be a commutative ring which satisfies the following axioms: (i) the minimal condition holds in each residue class ring $\mathfrak{J}|\mathfrak{A}$, $\mathfrak{A}$ an ideal in $\mathfrak{J}$, and (ii) the ring $\mathfrak{J}$ is integrally closed in its quotient field. The author proves that the Noether ideal theory holds in such a ring $\mathfrak{J}$. The proof depends on the observation of C. Hopkins [Duke Math. J. **4**, 664–667 (1938)] that a ring consisting of nilpotent elements and satisfying the minimal condition is nilpotent.

O. F. G. Schilling (Chicago, Ill.).

Neuhaus, Albert. Products of normal semi-fields. Trans. Amer. Math. Soc. **49**, 106–121 (1941). [MF 3668]

A semi-field is a commutative semi-simple algebra with a group of automorphisms whose order is the order of the algebra. It has been shown that a semi-field is the direct sum of isomorphic fields. A separable semi-field S of order n with a group of automorphisms G of order n is said to be normal with group G if a scalar extension of S contains n orthogonal idempotents permuted transitively by G. The author proves simply a result due to Teichmüller: A normal subgroup N of G leaves invariant a normal semi-field in S whose order is the index of N in G. The converse is shown to be false by a counter-example. It is shown that S has a normal basis. If S has group G and T has group G^*, then the direct product $S \times T$ has group $G \times G^*$. There will be

several direct product decompositions $S\times W$ of $S\times T$ if G has a normal subgroup N and G/N is isomorphic to a subgroup H_e^* of the center of G^*. A composition of these W's is defined which is associative if $G=N\times H_e$, and forms an abelian group if G is abelian. A crossed product of S and G is defined and interrelationships of crossed products and direct products are investigated. *M. Hall.*

Scholz, Arnold. Totale Normenreste, die keine Normen sind, als Erzeuger nichtabelscher Körpererweiterungen. II. J. Reine Angew. Math. **182**, 217–234 (1940). [MF 3438]

Let K be a finite normal extension of an algebraic number field K_0, with G the Galois group of K/K_0. If G is cyclic, a number α of K_0 is the norm $N\beta$ of a number β of K whenever α is a total norm residue (that is, whenever $\alpha\equiv N\beta$ (mod $\mathfrak{p}^e$) has a solution β for every e and every prime divisor $\mathfrak{p}$ of K_0). When G is abelian, Hasse showed that this result need no longer be true [Nachr. Ges. Wiss. Göttingen **1931**, 64–69]. The present paper studies the exact extent of the divergence, as measured by the knot $\mathfrak{K}$ of K, defined as the multiplicative factor group of the total norm-residues modulo the norms of K/K_0. A tool is the multiplicator M of G, introduced by Schur [J. Reine Angew. Math. **127**, 20–50 (1904)]. The essential result of the investigation asserts that the knot of the field K is always a homomorphic image of the multiplicator of its group G, and that for given G there exist fields K and K_0 with M itself as knot. In particular, a group G with the multiplicator 1 (a "closed" group) never gives rise to a proper knot.

The proof extends a device used in a previous paper by the author [J. Reine Angew. Math. **175**, 100–107 (1936)], realizing the knot $\mathfrak{K}$ as part of the Galois group of a central extension of K (a field $\Lambda\geqq K$ which is normal over K_0 is called a central extension of K if the group of Λ/K is in the center of the group of Λ/K_0). Specifically, K has a central extension Λ such that the group of Λ over the composite $K\Lambda^0$ is exactly the knot $\mathfrak{K}$, where Λ^0 is the maximal subfield of Λ abelian over K_0 [Theorem 2]. From this the previous result about the multiplicator may be derived, using a purely group-theoretic limitation of central extensions [Scholz, Monatsh. Math. Phys. **48**, 340–352 (1939); these Rev. **1**, 104]. Conversely, one may describe the group of all central extensions of K if one uses only the norms of those elements β of K with $\beta\equiv 1$ (mod $\mathfrak{t}$), for a suitable modulus $\mathfrak{t}$ of K, invariant under all automorphisms of K/K_0. The group of total norm residues for such elements modulo their norms is the ray-knot R, which has the original knot $\mathfrak{K}$ as a homomorphic image. Then [Theorem 5] for any central extension Λ of K the group of Λ over $K\Lambda^0$ is a homomorphic image of the ray-knot R, and there exist extensions Λ for which this homomorphism is an isomorphism. In this existence theorem one cannot always require that K contain the maximal abelian subfield Λ^0 of Λ. Conditions under which this supplementary restriction can be made are stated in terms of the absolute class field C of K and its maximal subfield C^0 abelian over K_0. If C^0 has an extension A abelian over K_0 such that the group of A/C^0 is isomorphic to the multiplicator M of the original group G, then there exists a central extension Λ with $\Lambda^0\leqq K$ and with R the group of Λ over K; indeed, one may choose for Λ any central extension which is maximal with the restriction that $\Lambda^0\leqq K$ [Theorem 5]. A similar result holds for the knot itself [Theorem 4]; if there exists an extension B of C^0 abelian over K_0 such that the group B/C^0 is the given knot $\mathfrak{K}$, then there exists, as in Theorem 2, a central extension Λ of K with $\Lambda^0\leqq K$ and with the group of Λ/K isomorphic to the knot $\mathfrak{K}$. By using a still different knot one may characterize those central extensions M/K for which each local field M_p is the composite of K_p with an abelian extension of K_{0p}. Since the central extensions are all abelian they may be treated over K by the devices of class field theory; this is the technique of the proofs. The theorems of the paper itself are couched in an elaborate special terminology which has been omitted in this summary.

S. MacLane (Cambridge, Mass.).
O. F. G. Schilling (Chicago, Ill.).

Dubisch, Roy. Non-cyclic algebras of degree four and exponent two with pure maximal subfields. Bull. Amer. Math. Soc. **47**, 131–133 (1941). [MF 3827]

The author proves that, if K is the field of all rational functions in two independent indeterminates, there exist non-cyclic normal division algebras of degree four and exponent two over K and containing an element generating a pure field of degree 4 over K. *A. A. Albert* (Chicago, Ill.).

Deuring, Max. Arithmetische Theorie der Korrespondenzen algebraischer Funktionenkörper. II. J. Reine Angew. Math. **183**, 25–36 (1940). [MF 4226]

Hasse has recently shown how abelian integrals can be treated as differentials of an algebraic function field K over any perfect coefficient field k, which may have prime characteristic [J. Reine Angew. Math. **172**, 55–63 (1935)]. This raises the problem of generalizing Abel's theorem on integrals of first kind. This is done in the present paper, for an algebraically closed field k; it uses the results of part I [J. Reine Angew. Math. **177**, 161–191 (1937)] which treated algebraic correspondences of K to itself by means of prime divisors in the composite field KL, obtained from K and an isomorphic image $L=K^\varphi$ (φ leaves constants fixed). A prime divisor P of KL/L is a homomorphic map of KL on a finite algebraic extension L^* of L. If P maps K homomorphically on k, this mapping is a prime divisor of K, and P is called constant. A non-constant P maps K isomorphically on a subfield K^τ of the residue class field L^*, and L^* is then the composite LK^τ. Each P defines a correspondence $d\rightarrow P(d)$ which is a homomorphism of the divisors d of L to the divisors of K. (Specifically, if P is non-constant, $P(d)^\tau=Nd$, N the norm from L^* to K^τ.) This correspondence induces a homomorphism (or "multiplicator") $\alpha=\alpha(P)$ of the class group of divisors of degree zero in L into the corresponding group in K.

If K has genus g, there are g linearly independent differentials du_i of first kind in K. If T denotes a "trace" from L^* to L, then, for each non-constant P, $T(du_i^\tau)$ is a differential of first kind in L, hence has an expression $T(du_i^\tau)=\sum m_{ij}dv_j$, where m_{ij} are constants in k, and the dv_j form a basis of the differentials of first kind in L. The g by g matrix $M(P)=\|m_{ij}\|$ corresponds to the matrix of a (classical) complex multiplication. Consider now any divisor $D=\Pi P_i^{r_i}$ of KL/L, where each factor P_i is non-constant. It determines a correspondence $D(d)=\Pi P_i(d)^{r_i}$ and hence a multiplicator $\alpha=\alpha(D)$ of the class group, as well as a matrix $M(D)=\sum r_iM(P_i)$. The set of all multiplicators α is a ring, in which addition corresponds to the product of divisors, multiplication to the multiplication of correspondences. The major result asserts that $\alpha(D)\rightarrow M(D)$ is a homomorphic representation of the ring of multiplicators by a matrix ring (for characteristic p it is not an isomorphism). Deuring also shows how to

deduce more simply results of Hasse on the structure of this ring in the elliptic case ($g=1$) [J. Reine Angew. Math. **175**, 193–208 (1936)], and proves that Rosati's antiautomorphism of this ring can be defined by interchanging K and L in the ring composite of the fields K and L. It is surprising to find no reference to Albert's investigations of these anti-automorphisms [Ann. of Math. (2) **36**, 886–964 (1935)].
S. MacLane (Cambridge, Mass.).

NUMBER THEORY

★Lehmer, Derrick Henry. Guide to Tables in the Theory of Numbers. Bulletin of the National Research Council, no. 105. National Research Council, Washington, D. C., 1941. xiv+177 pp.

A descriptive account is given [pp. 1–83] of existing tables in the theory of numbers; this is set forth in such a way as to indicate clearly what each table contains. A bibliography [pp. 85–125], arranged alphabetically by authors, gives exact references to the material cited and supplies information concerning the holdings, in libraries of the United States and Canada, of the books and pamphlets to which reference is made. Errata in the tables are listed [pp. 127–171], the sources being given in the cases of errata previously printed; Lehmer's contributions in the way of new indications of errata are notable. The book is indexed.
R. D. Carmichael (Urbana, Ill.).

★Candy, Albert L. Pandiagonal Magic Squares of Composite Order. Published by the author, Lincoln, Neb., 1941. x+155 pp. $1.00.

This book does not contain much that is essentially new. But the author admits he was "working chiefly for the fun of it." The style is lucid and the treatment elementary. As in his "Pandiagonal Magic Squares of Prime Order," 1940 [these Rev. 1, 290], each magic square is derived from the "natural square" by suitably permuting whole rows and columns and then re-writing the numbers in a new order, following some generalized "Knight's tour." According to Kraïtchik [La Mathématique des Jeux, Brussels, 1930, pp. 139–153], a similar method was invented by Margossian. The present author stresses the magnitude of the number of distinct pandiagonal squares of various orders that can be constructed by means of such simple rules; for example, the number of squares of order 16, of one particular type, is about 10^{30}.
H. S. M. Coxeter (Toronto, Ont.).

Moessner, Alfred. Verschiedene Diophantische Probleme und numerische Identitäten. Tôhoku Math. J. **47**, 188–200 (1940). [MF 4020]

The author gives certain algebraic identities which yield some solutions of various diophantine systems involving equal sums and products of like powers. For example, he points out that certain cubic polynomials in four variables satisfy the system $X_1+X_2+X_3=Y_1+Y_2+Y_3$; $X_1\cdot X_2\cdot X_3=Y_1\cdot Y_2\cdot Y_3$, identically.
I. A. Barnett.

Bang, A. S. Some algebraic identities. Mat. Tidsskr. B. **1940**, 62–65 (1940). (Danish) [MF 3881]

From the known identity
$$2(a^2+ab+b^2)^4=(a^4-b^4)^4+(a^2+2ab)^4+(2ab+b^2)^4$$
is derived for each n an identity of the form
$$2(a^2+ab+b^2)^{4n}=A^4+B^4+C^4,$$
where A, B, C are polynomials in a and b with integral coefficients.
B. Jessen (Copenhagen).

Bang, A. S. On integers of the form $a^x \pm b$, where a is prime to b. Mat. Tidsskr. B. **1940**, 21–24 (1940). (Danish) [MF 3878]

Bang, A. S. On integers representable as a sum of three or four cubes. Mat. Tidsskr. B. **1940**, 25–42 (1940). (Danish) [MF 3879]

Reitan, L. Some considerations on a problem in number theory. Norsk Mat. Tidsskr. **22**, 110–115 (1940). (Norwegian) [MF 3759]

Elementary considerations connected with the Diophantine equation $m^2=x^2+y^2+z^2$.
W. Feller.

Reitan, L. Euler's function and its number-tree. Norsk Mat. Tidsskr. **22**, 116–118 (1940). (Norwegian) [MF 3760]

The number-tree referred to in the title is a graph in which the joints represent integers and the branches join consecutive integers $m_1\rightarrow m_2\rightarrow m_3\rightarrow\cdots$ for which $m_i=\phi(m_{i+1})$. Thus the main branch is $2\rightarrow 4\rightarrow 8\rightarrow 16\rightarrow\cdots$; from 2 the branches $2\rightarrow 3$ and $2\rightarrow 6\rightarrow 18\rightarrow 54\rightarrow 81$ emanate, etc. This defines an "order" of integers. It is stated that there exist primes of arbitrary high order. The paper contains no new results.
W. Feller (Providence, R. I.).

Skolem, Th. Some notes on L. Reitan's papers. Norsk Mat. Tidsskr. **22**, 119–123 (1940). (Norwegian) [MF 3761]

This paper contains elementary proofs of some assertions made by Reitan [cf. the preceding two reviews] and some simple considerations on the "number-tree" of the preceding review.
W. Feller (Providence, R. I.).

Skolem, Th. Einige Sätze über Polynome. Avh. Norske Vid. Akad. Oslo. I. **1940**, no. 4, 16 pp. (1940). [MF 3783]

Let $f_i(x_1, \cdots, x_m)$, $i=1, \cdots, n$, be integral valued polynomials. The author has shown previously [Norske Vid. Selsk. Forh. **12**, no. 1, 1–4 (1939)] that, if the f_i have the greatest common divisor 1 for arbitrary integral values of the variables $x_1, \cdots, x_m$, there exist integral valued polynomials $F_i(x_1, \cdots, x_m)$ satisfying the identity
$$\sum_{i=1}^{n} f_i(x_1, \cdots, x_m)F_i(x_1, \cdots, x_m)=1.$$
The present paper considers such an identity under the weaker hypothesis that the f_i do not vanish simultaneously. In this case the F_i cannot be integral valued for integral values of $x_1, \cdots, x_m$ for which the f_i have a common divisor $d(x_1, \cdots, x_m)>1$. The author shows that the F_i may be chosen so that, for every integral m-tuple $x_1, \cdots, x_m$, the F_i have as common denominator $d(x_1, \cdots, x_m)$.
H. T. Engstrom (New Haven, Conn.).

Lehmer, D. H. A note on the linear Diophantine equation. Amer. Math. Monthly **48**, 240–246 (1941). [MF 4303]

A solution in integers x, y of the linear Diophantine equation $a_0x-a_1y=K$, where $0<a_1<a_0$ and $(a_0, a_1)=1$, may be found by writing a_0/a_1 as a continued fraction and com-

puting the convergents. The author points out that this method is twice as laborious as necessary and gives instead the following result. The equation $a_0x-a_1y=1$, where $0<a_1<a_0$ and $(a_0, a_1)=1$, has a solution $x=(-1)^n C_{n-1}$, $y=(-1)^n C_n$, where $C_0=0$, $C_1=1$, $C_{k+1}=q_{n-k}C_k+C_{k-1}$, and the q's are the partial quotients in the continued fraction development of a_0/a_1. It is noted that other sets of q's may be used, the only requirement being that the sequence terminate with $a_{n-1}=q_na_n$, $|a_n|=1$. *R. D. James.*

Wahlgren, Agne. Sur l'équation $ax^2+bxy+cy^2=es^2$. Ark. Mat. Astr. Fys. **27A**, no. 6, 26 pp. (1940). [MF 3495]

Let $f(x, y)$ be a positive definite quadratic form of discriminant D with relatively prime integer coefficients. The author gives a method for solving the Diophantine equation $f(x, y)=es^2$, using the composition of the representations of the powers of primes p by forms of discriminant D, where p is a divisor of D. [Cf. Dickson, History of the Theory of Numbers, vol. 2, 1934, pp. 404–407; Dickson, Introduction to the Theory of Numbers, pp. 44–48; E. T. Browne, Amer. Math. Monthly **42**, 502–503 (1935).] *A. Brauer.*

Bell, E. T. Note on a certain type of Diophantine system. Bull. Amer. Math. Soc. **47**, 155–159 (1941). [MF 3833]

The author gives necessary and sufficient conditions that the system of Diophantine equations $ax^2+bx=y^2$, $cx^2+dx=z^2$ shall have integral solutions. These conditions are that b and d be simultaneously representable in two forms of degree seven, and the solution of the original system is given in terms of the coefficients of these forms. The method is made to depend on the fact that the given system is equivalent to a pure multiplicative system [author's paper, Amer. J. Math. **55**, 50–56 (1933)]. The results are generalized to systems of s equations whose members involve certain polynomials. *I. A. Barnett* (Urbana, Ill.).

Brauer, Alfred. On a property of k consecutive integers. Bull. Amer. Math. Soc. **47**, 328–331 (1941). [MF 4186]

It is proved that if $k\geqq 300$ there exists a sequence of k consecutive integers such that no one of these k integers is relatively prime to the product of the others. For $17\leqq k\leqq 430$ this had already been proved by Pillai who had conjectured the general result [Proc. Indian Acad. Sci., Sect. A. **11**, 6–12 (1940); cf. these Rev. **1**, 199]. Pillai also proved that for $k<17$ there exists in every such sequence an integer relatively prime to all the rest. The proof uses an "elementary" estimate of the number of primes between x and $2x$ and is accomplished by constructing a sequence of k consecutive integers each of which is divisible by at least one of the primes less than $2[k/4]$. It follows at once that each number in the sequence has one of these primes in common with some other number in the set.

H. W. Brinkmann (Swarthmore, Pa.).

Cramer, G. F. On "almost perfect" numbers. Amer. Math. Monthly **48**, 17–20 (1941). [MF 3718]

The author defines an "almost perfect number" with respect to an arbitrary $\epsilon>0$ as a positive integer N such that the ratio $F(N)=S(N)/N$ differs from 2 by less than ϵ. Here $S(n)$ is used to denote the sum of the divisors of N. It is shown in an elementary way (using Bertrand's postulate) that for each $\epsilon>0$ there exist infinitely many almost perfect numbers. In fact it is shown that infinitely many odd N's exist for which $F(N)$ differs from any fixed real number $A>1$ by less than ϵ. A similar theorem is true for even N's provided $A>2$. *D. H. Lehmer.*

Pipping, Nils. Goldbachsche Spaltungen der geraden Zahlen x für $x=60000-99998$. Acta Acad. Aboensis **12**, no. 11, 18 pp. (1940). [MF 3997]

Goldbach's conjecture that every even integer not less than 6 is representable as a sum of two odd primes has been verified for all even integers in the interval $6\leqq x<60{,}000$. The author extends this verification to all even integers in the interval $60{,}000\leqq x<100{,}000$ using the following method. Let q be the largest prime not greater than $x-3$. It can easily be determined from existing tables of primes whether or not $x-q$ is also a prime. If it is then x is a sum of two odd primes. For all even x in the above interval such that $x-q$ is not a prime the author finds and tabulates the least prime m_x for which $x-m_x$ is also a prime. Again x is a sum of two odd primes. *R. D. James* (Saskatoon, Sask.).

Wall, H. S. A continued fraction related to some partition formulas of Euler. Amer. Math. Monthly **48**, 102–108 (1941). [MF 3913]

Using continued fractions the author finds a new and ingenious derivation of a number of partition formulas of Euler. *W. Leighton* (Houston, Tex.).

Dyer-Bennet, John. A note on partitions of the set of positive integers. Amer. Math. Monthly **48**, 15–17 (1941). [MF 3717]

This paper deals with the problem of separating the positive integers into classes of equivalent numbers so as to preserve homomorphisms with respect to addition, multiplication and "exponentiation." More definitely, if we denote the equivalence relation by $\sim$, the problem is to find all cases in which $a\sim b$, $c\sim d$ imply (1) $a+c\sim b+d$; (2) $ac\sim bd$; (3) $a^c\sim b^d$. For (1) it is shown that the equivalence must be equality for the numbers $1, 2, 3, \cdots, i-1$, while for larger numbers it is the relation of congruence modulo m, where m and i are unrelated. The separation of the integers into classes by such an equivalence is called a "partition of type (i, m)." For such a partition (2) also holds. In order that (1), (2) and (3) hold, it is necessary and sufficient to relate i and m so that m is divisible by the $(i+1)$st power of no prime and to so choose m that if p be any prime factor of m then m is also divisible by $p-1$. Partitions of type $(1, m)$ for which (3) holds were discussed in a previous paper [Amer. Math. Monthly **47**, 152–154 (1940); these Rev. **1**, 201] and it was shown that m has, in this case, only 5 values. It is conjectured that in general for any fixed i there are only a finite number of partitions of type (i, m) for which (3) holds. *D. H. Lehmer.*

Kuhn, Pavel. Eine Formel für die Summe der Möbiusfaktoren. Norske Vid. Selsk. Forh. **13**, 112–114 (1940). [MF 3870]

A formula for the sum $\sigma(x)=\sum_{n=1}^{x}\mu(n)$ is transformed to obtain a formula for $\sigma(x)-\sigma(x/2)$ which could be used to compute values of $\sigma(x)$. *H. S. Zuckerman.*

Gupta, Hansraj. On a table of values of $L(n)$. Proc. Indian Acad. Sci., Sect. A. **12**, 407–409 (1940). [MF 3764]

This note gives data on the behavior of the function $L(x)=\sum_{m\leqq x}\lambda(m)$, where $\lambda(m)$ is Liouville's purely multiplicative function for which $\lambda(p^n)=(-1)^n$, p a prime. Thus $L(x)$ is the excess of the number of those integers not exceeding x which have an even number of prime factors over the number of those having an odd number of prime factors (e.g., $2^3\cdot 3^2\cdot 7$ has 6 prime factors). For $x\geqq 2$, $L(x)\leqq 0$ ac-

cording to a conjecture of Pólya, who verified this fact for $x \leqq 1500$ [Jber. Deutsch. Math. Verein. **28**, 38–40 (1919)]. The author has extended this verification to $x \leqq 20000$, and publishes three small tables giving data as follows. For each integer t from 0 to 10, the first table gives the greatest value of $n \leqq 20000$ for which $L(n) = -t$. Only the first two or three entries in this table are of much interest, since $L(48512) = -2$. The greatest n for which $L(n)$ is known to be zero is 586. The second table gives for each $k \leqq 150$ the least value of n for which $L(n) = -k$. For $n \leqq 20000$, $L(n)$ has a minimum of -150 at $n = 15810$. The third table gives 19 extreme values of $L(n)$ for $n \leqq 20000$. For this range the function $L^2(n)/n$ is greatest for 9840, where it has the value 1.66504, from which it appears that $L(x) = O(x^{1/2})$.

D. H. Lehmer (Berkeley, Calif.).

Wintner, Aurel. On Dirichlet's divisor problem. Proc. Nat. Acad. Sci. U. S. A. **27**, 135–137 (1941). [MF 3809]

Let $D(x) = d(1) + d(2) + \cdots + d([x])$, where $d(n)$ is the number of divisors of n; and $F(x) = x(\log x + 2C - 1)$. Let $\bar{D}(x)$ be the limit of $\frac{1}{2}\{D(x+h) + D(x-h)\}$ as $h \to 0$. The function $Q(t) = \{\bar{D}(t^2) - F(t^2)\}/t^{\frac{1}{2}}$ is said to be almost periodic (B^2), $1 \leqq t < \infty$, and to have the Fourier series

$$Q(t) \sim \frac{1}{2^{\frac{1}{2}}\pi} \sum_{n=1}^{\infty} \frac{d(n)}{n^{\frac{3}{4}}} \cos(4\pi n^{\frac{1}{2}} t - \pi/4).$$

The proof is to be published elsewhere. *G. Pall.*

Behrend, F. A. On obtaining an estimate of the frequency of the primes by means of the elementary properties of the integers. J. London Math. Soc. **15**, 257–259 (1940). [MF 4036]

Using only elementary properties of integers, T. S. Broderick proved [J. London Math. Soc. **14**, 303–310 (1939); cf. these Rev. **1**, 41] that $1/20 < \pi(n)(\lambda(n)/n) < 14$ $(n \geqq 2)$, where $\pi(n) = \sum_{p \leqq n} 1$, $\lambda(n) = \sum_{\nu=1}^{n} 1/\nu$. In a simpler manner by working with 2^{k+1} instead of n and using the idea of the proof of Theorem 112 in Landau's Vorlesungen über Zahlentheorie, vol. I, the author shows that $1/8 < \pi(n)(\lambda(n)/n) < 12$ $(n \geqq 2)$ and indicates that $\Lambda(n) = \lambda(n) - 1$ is the more natural function to use. *R. D. James* (Saskatoon, Sask.).

Selberg, Sigmund. Über die zahlentheoretische Funktion $\pi_n(x)$. Norske Vid. Selsk. Forh. **13**, 30–33 (1940). [MF 3868]

The formula

$$\pi_n(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{n-1}}{(n-1)!},$$

where $\pi_n(x)$ is the number of positive square-free integers not greater than x which are products of n primes, is proved by means of a consideration of the functions $\pi_{\nu,n}(\xi, x)$ which designate the number of positive square-free integers not greater than x which are products of ν primes not greater than ξ and $n-\nu$ primes greater than ξ. These methods can be used to obtain more refined asymptotic formulas involving $\pi_n(x)$. *H. S. Zuckerman* (Seattle, Wash.).

Selberg, Sigmund. Bemerkung zu einer Arbeit von Viggo Brun über die Riemannsche Zetafunktion. Norske Vid. Selsk. Forh. **13**, 17–19 (1940). [MF 3866]

The author proves the formula

$$\zeta(s) = \frac{1}{s-1} + \sum_{m=2}^{\infty} (-1)^m \frac{2^{s-1} + 4^{s-1} + \cdots + 2^{k(s-1)}}{m^s},$$

where $2^k \leqq m < 2^{k+1}$, which has been announced by Viggo Brun [9. Congr. Math. Scand., Helsingfors, 1938, pp. 101–104]. *H. S. Zuckerman* (Seattle, Wash.).

Ingham, A. E. On the estimation of $N(\sigma, T)$. Quart. J. Math., Oxford Ser. **11**, 291–292 (1940). [MF 3791]

Let $N(\sigma, T)$ be the number of zeros $\rho = \beta + \gamma^i$ of the Riemann zeta-function for which $\beta \geqq \sigma$, $0 < \gamma \leqq T$. It is proved that $N(\sigma, T) = O(T^{\lambda(\sigma)(1-\sigma)} \log^5 T)$ uniformly for $\frac{1}{2} \leqq \sigma \leqq 1$ as $T \to \infty$ with (A) $\lambda(\sigma) = 3/(2-\sigma)$. The author has previously shown [Quart. J. Math., Oxford Ser. **8**, 255–266 (1937)] that the estimate is valid with (B) $\lambda(\sigma) = 1 + 2\sigma$ and with (C) $\lambda(\sigma) = 2 + 4c$, where c is a constant for which $\zeta(\frac{1}{2} + t_i) = O(t^c)$. (A) is an improvement on (B) for $\frac{1}{2} \leqq \sigma \leqq 1$, while (C) is the best near $\sigma = 1$. The estimate of the difference between consecutive primes is not changed because it depends on the maximum of $\lambda(\sigma)$ for $\frac{1}{2} \leqq \sigma \leqq 1$ and this is still given by (C). The proof of (A) is so similar to that of (B) and (C) that the author merely notes the necessary changes. *H. S. Zuckerman* (Seattle, Wash.).

Rademacher, Hans and Whiteman, Albert. Theorems on Dedekind sums. Amer. J. Math. **63**, 377–407 (1941). [MF 4159]

Let $((x)) = x - [x] - \frac{1}{2}$ for x not an integer and $((x)) = 0$ for x an integer. Then, in the notation of this paper, the Dedekind sums are defined to be $s(h, k) = \sum_{\mu=1}^{k}((\mu/k))((h\mu/k))$. These sums were used by Dedekind in his "Erläuterungen zu den Riemannschen Fragmenten über die Grenzfälle der elliptischen Modulfunktionen." They are intimately connected with the transformation formula for $\log \eta(\tau)$, where $\eta(\tau) = e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau})$ and where τ is subjected to a modular transformation. The formulas stated by Dedekind were either without proof or were proved by function-theoretic methods. The present paper supplies purely arithmetic proofs for Dedekind's formulas and it includes some new ones. A particularly simple proof of the reciprocity formula $12s(h, k) + 12s(k, h) = -3 + h/k + k/h + 1/hk$, $h > 0$, $k > 0$, $(h, k) = 1$, is given. One section is devoted to verifying that the transformation formulas given by Riemann are actually equivalent to those obtained by Dedekind. The final part of the paper considers congruences involving $s(h, k)$. The principal theorem has to do with three variables. If a, b, c are positive integers, relatively prime in pairs, and if $24 | abc$ then

$$\{s(ab, c) - ab/12c\} + \{s(bc, a) - bc/12a\} - \{s(b, ac) - b/12ac\} \equiv 0 \pmod 2.$$

This theorem, together with two less general ones which take up cases where $24 \nmid abc$, lead to formulas for the factorization of

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \exp(\pi i s(h, k) - 2\pi i(hn/k)),$$

which appear in the formula for the number $p(n)$ of unrestricted partitions of n. These factorization formulas are of the same type as the multiplication theorems of D. H. Lehmer [Trans. Amer. Math. Soc. **43**, 271–295 (1938)].

H. S. Zuckerman (Seattle, Wash.).

Mardjanichvili, C. et Segal, B. Sur une estimation des sommes de Weyl. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 731–734 (1940). [MF 3573]

Let $n \geqq 2$; $\alpha, \alpha_1, \cdots, \alpha_n$ be real numbers, $f(x) = \alpha x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n$; Q and P $(\geqq 3)$ be integers. J. G. van der Corput

[Nederl. Akad. Wetensch., Proc. **42**, 461–467 (1939)] obtained an excellent upper bound for the Weyl sum $S=|\sum e^{2\pi i f(x)}|$ (summed over $x=Q+1, \cdots, Q+P$), involving an undetermined constant $c=c(n, \epsilon)$. A similar result is here derived, with an explicit c; it is proved that

$$|S|^{2^{n-1}}<11(l-1)(4P)^{2^{n-1}}\mu^{\sigma}((\Lambda+q/P)(1/q+1/P^{n-1}))^{1-\epsilon},$$

where $\alpha=a/q+\lambda/q^2$, $q>0$, $(a, q)=1$, $|\lambda|\leqq\Lambda$, $\Lambda\geqq 1$; $0<\epsilon\leqq\frac{1}{2}$; l is the least integer not less than $1/\epsilon$, $\mu=\log P^{n-1}+(n-1)^l-1$, and $l\sigma=(n-1)^l-1$. If $n\geqq 3$, the factor $11(l-1)$ can be replaced by 8; if $f(x)$ is replaced by $mf(x)$, where m is an integer not less than 1, Λ is replaced in the right member by Λm. Use is made of an upper bound due to Mardjanichvili [same C. R. **22**, 387–389 (1939)], for $\tau_k{}^l(1)+\cdots+\tau_k{}^l(n)$, where $\tau_k{}^l(h)$ is the number of solutions of $x_1x_2\cdots x_k=h$.

G. Pall (Princeton, N. J.).

Overing, A. C. M. On entire functions. Bol. Mat. **13**, 260–268 (1940). (Spanish) [MF 3348]

The elementary properties of well-known multiplicative arithmetical functions are derived in an interesting way.

G. Pall (Princeton, N. J.).

Beiler, A. H. A peculiar property of the primitive roots of 13. Amer. Math. Monthly **48**, 185–187 (1941). [MF 4134]

Tschebotareff, N. G. A problem of the theory of algebraic numbers. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 283–290. (Russian) [MF 3521]

Let k be an algebraic field with group $\mathfrak{g}$, and let $\mathfrak{G}$ be an abstract group with an abelian normal subgroup $\mathfrak{H}$ such that $\mathfrak{G}/\mathfrak{H}$ is isomorphic with $\mathfrak{g}$. In an earlier article [J. Reine Angew. Math. **167**, 98–121 (1932)] the author reduced the existence of a field K containing k and having a group isomorphic with $\mathfrak{G}$ to the problem of existence of l-primary prime ideals $\mathfrak{p}$ for which the symbol $\{\pi_2\pi_3\cdots\pi_k|\mathfrak{p}\}$ has a given value; here $\pi_2, \cdots, \pi_k$ are the conjugates $\mathfrak{p}_2, \cdots, \mathfrak{p}_k$ of $\mathfrak{p}(=\mathfrak{p}_1)$ multiplied by certain lth powers of ideals. The problem of finding such prime ideals is now solved in the case $l=2$, for imaginary quadratic fields $K=k(\sqrt{-m})$, where m is a product of distinct odd primes of the form $4n+1$. The symbol $\{a-b\sqrt{-m}|a+b\sqrt{-m}\}$ is evaluated in various cases in which a^2+mb^2 is an odd prime.

G. Pall.

Humbert, Pierre. Note relative à l'article: Sur les nombres de classes de certains corps quadratiques. Comment. Math. Helv. **13**, 67 (1940). [MF 3989]

Acknowledgment that T. Nagell had proved in 1921 [Abh. Math. Sem. Hansischen Univ. **1**, 140] one of the theorems in the author's paper in Comment. Math. Helv. **12**, 233–245 (1939) [see these Rev. **2**, 39].

H. W. Brinkmann.

Rosser, Barkley. An additional criterion for the first case of Fermat's last theorem. Bull. Amer. Math. Soc. **47**, 109–110 (1941). [MF 3820]

In a previous paper [Bull. Amer. Math. Soc. **46**, 299–304 (1940); these Rev. **1**, 292] it was shown that if p is an odd prime and $a^p+b^p+c^p=0$ has a solution in integers prime to p then $m^{p-1}\equiv 1 \pmod{p^2}$ for each prime $m\leqq 41$. From this it followed that $p>41{,}000{,}000$. In the present paper the result is extended by similar methods to $m\leqq 43$ thus increasing the lower bound for p.

R. D. James (Saskatoon, Sask.).

Lehmer, D. H. and Lehmer, Emma. On the first case of Fermat's last theorem. Bull. Amer. Math. Soc. **47**, 139–142 (1941). [MF 3829]

A number is called an "A_n number" if it is divisible by no prime exceeding the nth prime p_n. Let $\phi_n(x)$ and $\phi_n{}^*(x)$ denote, respectively, the number of A_n numbers less than x and the number of odd A_n numbers less than x. If $a^p+b^p+c^p=0$ has a solution in integers prime to p and if $m^{p-1}\equiv 1 \pmod{p^2}$ for each prime $m\leqq p_n$, it is shown that $\phi_n(p^2/3)+\phi_n{}^*(p^2/3)\leqq(p-1)/2$. In order to apply this inequality, polynomials $P_n(x)$, $P_{n-1}^*(x)$ of degree n and $n-1$, respectively, giving lower bounds for $\phi_n(10^x)$ and $\phi_n{}^*(10^x)$ have been constructed by D. H. Lehmer [Duke Math. J. **7**, 341–353 (1940); these Rev. **2**, 149]. Rosser has shown that $m^{p-1}\equiv 1 \pmod{p^2}$ holds for each prime $m\leqq 43=p_{14}$ [see the preceding review]. The polynomials $P_{14}(x)$ and $P_{13}{}^*(x)$ are then calculated and it is found that the inequality $P_{14}(\log p^2/3)+P_{13}{}^*(\log p^2/3)\leqq(p-1)/2$ is satisfied only if $p\geqq 253{,}747{,}889$. Hence there is no solution of $a^p+b^p+c^p=0$ in integers prime to p if p is an odd prime $<253{,}747{,}889$.

R. D. James (Saskatoon, Sask.).

Whiteman, Albert Leon. Additive prime number theory in real quadratic fields. Duke Math. J. **7**, 208–232 (1940). [MF 3399]

Let θ denote the upper bound, known to be not greater than 1, of the real parts of all the zeros of the Hecke $\zeta(s, \lambda)$-functions in a real quadratic field $k=k(d^{\frac{1}{2}})$. Let μ denote a totally positive (μ and its conjugate μ' both positive) algebraic integer in k divisible by all prime ideals of norm 2. It is proved that the number $P(V, V')$ of μ such that $0<\mu\leqq V$ and $0<\mu'\leqq V'$, which cannot be represented as a sum of two totally positive primes in k, satisfies $P(V, V')/(VV')^{2\theta-\frac{3}{2}+\epsilon}\to 0$ as $V, V'\to\infty$. Hence if $\theta<\frac{3}{4}$, almost all totally positive "even" integers in k are sums of two such primes. H. Rademacher had proved, under a similar hypothesis, that every large "odd" number in an algebraic field is a sum of three "primes" [Abh. Math. Sem. Hansischen Univ. **3**, 109–163 and 331–378 (1924); Math. Z. **27**, 321–426 (1926)]. It is pointed out that "the tools necessary to carry over Vinogradow's intricate analysis," and thus to eliminate the assumption $\theta<\frac{3}{4}$, "have yet to be created." The function $A_m(\nu)=\sum\log N(\omega_1)\cdots\log N(\omega_m)$, where $\omega_1, \cdots, \omega_m$ run over totally positive integers whose sum is ν, has, if $m\geqq 3$, the asymptotic expression

$$E=d^{\frac{1}{2}}(2h\log\eta)^{-m}\Gamma^{-2}(m)N(\nu)^{m-1}\mathfrak{S}_m(\nu),$$

where h is the class number and η the fundamental unit of k, and $\mathfrak{S}$ can be expressed by the product

$$\prod_{\mathfrak{P}|\nu}\left(1-\left(\frac{-1}{N(\mathfrak{P})-1}\right)^m\right)\prod_{\mathfrak{P}|\nu}\left(1+\frac{(-1)^m}{(N(\mathfrak{P})-1)^{m-1}}\right).$$

If $m=2$, an upper bound is obtained for $\sum(A_m(\mu)-E)^2$ from which the results stated above follow.

G. Pall.

Mardjanichvili, C. Sur un problème additif de la théorie des nombres. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 193–214 (1940). (Russian. French summary) [MF 2736]

The author considers the problem of the representation of n integers $N_1, \cdots, N_n$ ($N_i<N_{i+1}$) in the form

$$(1)\qquad N_\kappa=\sum_{\nu=1}^{s}p_\nu{}^\kappa,\qquad \kappa=1, \cdots, n,$$

where $p_1, \cdots, p_s$ are primes. Let $I(N_1, \cdots, N_n; s)$ be the

number of ordered sets $(p_1, \cdots, p_s)$ of primes satisfying (1). Then (Theorem I),

$$I(N_1, \cdots, N_n; s) = B(h_1, \cdots, h_{n-1}; s) N_n^{s/n-(n+1)/2} (\lg N_n)^{-s} \times \mathfrak{S}(N_1, \cdots, N_n; s) + O(N_n^{s/n-(n+1)/2} (\lg N_n)^{-s-\omega}).$$

Here,

$$\mathfrak{S} = \sum_{q_1, \cdots, q_n=1}^{\infty} A(q_1, \cdots, q_n; s; N_1, \cdots, N_n),$$

$$A(q_1, \cdots, q_n; s; N_1, \cdots, N_n) = \sum_{a_1, \cdots, a_n}{}' D^s \exp\left(-2\pi i \sum_{i=1}^{n} \frac{a_i}{q_i} N_i\right),$$

$$D(a_1, q_1; \cdots; a_n, q_n) = \frac{1}{\phi(q_1 \cdots q_n)} \sum_r{}' \exp\left(2\pi i \left(\frac{a_n}{q_n} r^n + \cdots + \frac{a_1}{q_1} r\right)\right),$$

$$N_\kappa = h_\kappa N_n^{\kappa/n}, \qquad \kappa = 1, \cdots, n,$$

ω is a positive constant, and $0 < C_1(n, s) \leqq B(h_1, \cdots, h_{n-1}; s) \leqq C_2(n, s)$. In the last sum the summation is extended over a reduced system of residues mod $q_1 \cdots q_n$, whereas in the second sum each a_i runs over the reduced system of residues mod q_i. The above estimate for I holds under the assumption that $s \geqq 5n(n+1)(n+2) \lg n$ and that $N_1, \cdots, N_n$ have the following property: there exists $\epsilon > 0$ such that for a suitably chosen set of numbers h_κ' for which $h_\kappa - \epsilon \leqq h_\kappa' \leqq h_\kappa$, the system $\xi_1^\kappa + \cdots + \xi_s^\kappa = h_\kappa$ $(\kappa = 1, \cdots, n)$ has positive solutions ξ_ν. Next (Theorem II) the author proves that, for n integers $M_1, \cdots, M_n$ satisfying a certain system of determinantal congruences, there exists an $s_0 \leqq \beta_0(n)$ (here β_0 depends on n alone) and such that

$$\mathfrak{S}(M_1, \cdots, M_n; s_0) \geqq C_0(n) > 0.$$

In the proofs use is made of certain estimates of trigonometric sums by Vinogradow [Rec. Math. [Mat. Sbornik] N.S. **2** (44), 179–194 (1937); N.S. **3** (45), 435–471 (1938), Lemma 6; C. R. (Doklady) Acad. Sci. URSS (N.S.) **17**, no. 4 (1938)] and Mordell [Quart. J. Math., Oxford Ser. **3** (1932)]. Combining the results of the first two theorems the author obtains (Theorem III) conditions on $N_1, \cdots, N_n$ under which (1) would have prime solutions $p_1, \cdots, p_s$ with the bound of s depending only on n. *A. E. Ross.*

Archibald, Ralph G. Waring's problem: squares. Scripta Math. **7**, 33–48 (1940). [MF 4129]

This article is an historical account of the problem of representing an integer as a sum of two, three or four squares. It closes with a reproduction of the proof in Landau's Vorlesungen über Zahlentheorie, vol. I, pp. 107–109, that every positive integer is a sum of four squares.

R. D. James (Saskatoon, Sask.).

Sugar, Alvin. On a result of Hua for cubic polynomials. Bull. Amer. Math. Soc. **47**, 164–165 (1941). [MF 3835]

It is proved neatly that every integer is a sum in infinitely many ways of five values of $f(x) = \epsilon(x^3 - x)/6 + kx + c$ for integers x; and a sum of four values of $(x^3 - x)/6 + kx + c$. Here ϵ, c, k are given integers, k prime to ϵ. Hua's result [Tôhoku Math. J. **41**, 361–366 (1936)] involved seven summands, $k = 1$. The proof: $f(x) \equiv kx + c \pmod{\epsilon}$, $f(x) \equiv n - 4c \pmod{\epsilon}$ is solvable, $n = m\epsilon + 4c + f(t)$, $m\epsilon + 4c = f(m+1) + f(m-1) + 2f(-m)$ identically; if $\epsilon = 1$ take $m = n - 4c$. *G. Pall.*

Tornheim, Leonard. Linear forms in function fields. Bull. Amer. Math. Soc. **47**, 126–127 (1941). [MF 3825]

Let $F(z)$ be the field of rational functions in z over F. Suppose that V is the valuation belonging to the pole of z. Then define the degree of an element r of $F(z)$ as $-V(r)$. The author proves the following analogue of Minkowski's theorem: "Let $L_i = \sum_{j=1}^{n} a_{ij} x_j$, $i = 1, \cdots, n$, be n linear expressions with coefficients a_{ij} in $F(z)$ and with the determinant $|a_{ij}|$ of degree d. Then for any set of n integers $c_1, \cdots, c_n$ which satisfy the condition $\sum_{i=1}^{n} c_i > d - n$ there exists a set of values for $x_1, \cdots, x_n$ in $F[z]$ and not all zero such that each L_i has degree at most c_i." For the proof it suffices to consider the case when all c_i are equal. Furthermore, the elements a_{ij} may be taken as polynomials in z. Due to the validity of the Euclidean algorithm in $F[z]$ the matrix (a_{ij}) can be transformed into triangular form. The theorem is then readily proved by considering vector spaces of n-tuples of polynomials of bounded degree.

O. F. G. Schilling (Chicago, Ill.).

Pall, Gordon. Simultaneous representation in a quadratic and linear form. Duke Math. J. **8**, 173–180 (1941). [MF 3952]

The solvability in integers of the simultaneous equations

$$c_1 x_1^2 + \cdots + c_s x_s^2 = a, \quad c_1 x_1 + \cdots + c_s x_s = b \tag{1}$$

is considered. Put $t = c_1 + \cdots + c_s$. The identity

$$ta - b^2 = \sum_{i<k} c_i c_k (x_i - x_k)^2 \tag{2}$$

and the substitution $y_j = x_1 - x_j$, $j = 2, \cdots, s$ leads to a representation of $ta - b^2$ by a quadratic form in the $s-1$ y's. A study of this single form leads to criteria for the solvability of the system (1), including conditions for the existance of non-negative solutions. Various examples and a table of forms with $s = 4$ are given. *M. Hall.*

Hecke, E. Über die Darstellung der Determinante einer positiven quadratischen Form durch die Form. Viertel-jschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 64–70 (1940). [MF 3484]

Let $Q(x_1, \cdots, x_{2k}) = Ax_1^2 + Bx_1x_2 + \cdots$ be a positive quadratic form with $2k$ variables and integral coefficients $A, B, \cdots$ and the determinant of the matrix of $2Q$ an odd prime number q. The theta function

$$\vartheta(\tau) = \sum_{(n)} e^{2\pi i \tau Q(n_1, \cdots, n_{2k})} = \sum_{n=0}^{\infty} a(n, Q) e^{2\pi i n \tau}$$

is an entire modular form and $\vartheta(\tau') = \chi(d)(c\tau + d)^k \vartheta(\tau)$ for any modular substitution $\tau' = (a\tau + b)/(c\tau + d)$ with $c \equiv 0 \pmod{q}$, where $\chi(d) = (d/q)$ is Legendre's symbol. The linear set of all functions $F(\tau)$ with those properties, the type $(-k, q, \chi)$, is transformed into itself by certain operations T_m which have been studied by the author in previous papers [Math. Ann. **114**, 1–28 and 316–351 (1937); $T_m = T(m)$ of next review]. A number λ_m is called a characteristic value of T_m if a function F of the given type exists which is transformed by T_m into $\lambda_m F$. These λ_m appear in the multiplicative laws for the representation numbers $a(n, Q)$; in particular, λ_q is connected with the number of representations of q by the quadratic form Q. It is proved that λ_q has the absolute value $q^{(k-1)/2}$. The examples $k = 2$, $q = 29$ and $k = 3$, $q = 19$ are investigated. *C. L. Siegel* (Princeton, N. J.).

Hecke, E. Analytische Arithmetik der positiven quadratischen Formen. Danske Vid. Selsk. Math.-Fys. Medd. **17**, no. 12, 134 pp. (1940). [MF 3346]

Let $Q = Q(x_1, x_2, \cdots, x_{2k}) = Ax_1^2 + Bx_1x_2 + \cdots$ be a positive quadratic form of $2k$ variables with integral coefficients $A, B, \cdots$. If $n_1, \cdots, n_{2k}$ run over all integers and $a(n, Q)$

denotes the number of representations of n by Q, the function

$$\vartheta(\tau, Q)=\sum_{(n)} e^{2\pi i\tau Q(n_1, \cdots, n_{2k})}=\sum_{n=0}^{\infty} a(n, Q)e^{2\pi i n\tau}$$

is an entire modular form of dimension $-k$ which is invariant under a congruence subgroup of the modular group. There exists for any given Q a certain positive integer N and a character $\epsilon(d)$ modulo N such that (1) $(c\tau+d)^{-k}\vartheta(\tau', Q)=\epsilon(d)\vartheta(\tau, Q)$ for all modular substitutions $\tau'=(a\tau+b)/(c\tau+d)$ with $c\equiv 0 \pmod N$. Consider now all entire modular forms $F=F(\tau)$ of the type $(-k, N, \epsilon)$, that is, modular forms of dimension $-k$ satisfying (1); they are a linear set with a finite basis $F_1, \cdots, F_\kappa$. Let m be an integer relatively prime to N. If α runs over all positive divisors of $m=\alpha\delta$ and β over a complete system of residues modulo α, the linear operation

$$F\rightarrow F|T(m)=m^{-1}\sum_{\alpha,\beta}\epsilon(\alpha)\alpha^k F\left(\frac{\alpha\tau+\beta}{\delta}\right)$$

transforms the set $(-k, N, \epsilon)$ into itself and satisfies the law of composition

$$T(m_1)T(m_2)=\sum_{d|(m_1,m_2)} T\left(\frac{m_1m_2}{d^2}\right)\epsilon(d)d^{k-1}.$$

Hence

$$F_\rho|T(m)=\sum_{\sigma=1}^{\kappa}\lambda_{\rho\sigma}(m)F_\sigma, \qquad \rho=1, \cdots, \kappa,$$

and the matrices $\lambda(m)=(\lambda_{\rho\sigma}(m))$ have also the property

$$\lambda(m_1)\lambda(m_2)=\sum_{d|(m_1,m_2)}\lambda\left(\frac{m_1m_2}{d^2}\right)\epsilon(d)d^{k-1}$$

and are therefore commutative. It follows from the results of Petersson [Math. Ann. 116, 401–412 (1939) and 117, 39–64 (1939); these Rev. 1, 294] that the matrices $\lambda(m)$ can be transformed simultaneously into the diagonal form. Consider now the characteristic functions of all the operators $T(m)$, that is, the solutions $F=F(\tau)$ of $F|T(m)=\omega_m F$ $(m=1, 2, \cdots)$ with constant ω_m. If $F(\tau)=\sum_{n=0}^{\infty}a_n e^{2\pi i n\tau}$, then the Dirichlet series $\phi(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ can be written as an infinite product

$$\phi(s)=\prod_p (1-a_p p^{-s}+\epsilon(p)p^{k-1-2s})^{-1}$$

extended over all prime numbers p. There exist κ linearly independent Dirichlet series $\phi_1, \cdots, \phi_\kappa$ corresponding to characteristic functions and the Dirichlet series for any modular form of the type $(-k, N, \epsilon)$ is a linear combination of the κ Euler products $\phi_1, \cdots, \phi_\kappa$. The application of this result to the Dirichlet series

$$\sum_{n=1}^{\infty} a(n, Q)n^{-s}=\sum_{(n)}{}' Q(n_1, \cdots, n_{2k})^{-s}$$

gives the generalization of the well-known formula connecting the zeta-functions of the classes of ideals in a quadratic field with the zeta-functions containing the characters of the class-group.

A number of important examples are completely discussed. In these particular cases, the system of all theta-functions $\vartheta(\tau, Q)$ of the same type already gives the basis of an invariant linear set with respect to the operation $T(m)$, and the corresponding representation numbers $a(n, Q)$ are connected by certain multiplicative laws not containing other arithmetical functions than the $a(n, Q)$ alone. It is not known, however, if the $\vartheta(\tau, Q)$ of the same type always are the basis of a closed set for $T(m)$ in the general case. Any further progress seems to depend upon the discovery of new arithmetical notions generalizing the arithmetic of quaternions and adapted to the general quadratic form with an even number of variables. *C. L. Siegel.*

Mahler, Kurt. On a property of positive definite ternary quadratic forms. J. London Math. Soc. 15, 305–320 (1940). [MF 4041]

Some time ago Davenport gave a simple proof of the following result of Remak [J. London Math. Soc. 14, 47–51 (1939)]: Let $f(x, y, z)$ be a positive definite ternary quadratic form of determinant 1 which assumes its minimum in three linearly independent lattice points. Then given any three real numbers x_0, y_0, z_0, there are three integers x^1, y^1, z^1 such that

$$f(x_0+x^1, y_0+y^1, z_0+z^1)\leqq 3/4,$$

with equality if and only if $f(x, y, z)=x^2+y^2+z^2$ and $2x_0, 2y_0, 2z_0$ are odd integers. The author proves the following result which is shown to be more general than that of Davenport-Remak: Let

$$f(x, y, z)=a(x^2+y^2+z^2)+2bxy+2cxz+2dyz$$

be a positive definite ternary quadratic form of determinant 1 with coefficients satisfying the inequalities

$$1\leqq a\leqq 2^{1/3}, \quad 0\leqq b\leqq \tfrac{1}{2}a.$$

Then given any three real numbers x_0, y_0, z_0 there are three real numbers x_1, y_1, z_1, such that $x_1\equiv x_0 \pmod 1$, $y_1\equiv y_0 \pmod 1$, $z_1\equiv z_0 \pmod 1$, $f(x_1, y_1, z_1)\leqq 3/4$, with equality if and only if $x_0\equiv y_0\equiv z_0\equiv \frac{1}{2} \pmod 1$, $f(x, y, z)=x^2+y^2+z^2$. The method of the proof is geometrical and has been used by the author in a previous paper for the study of Hermitian forms [J. London Math. Soc. 15, 213–236 (1940); these Rev. 2, 148]. *P. Erdös* (Philadelphia, Pa.).

Sominski, I. S. Construction of the fundamental and basic domains of the arithmetic group of the automorphisms of an indefinite ternary quadratic form. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 10, 148–153 (1940). (Russian) [MF 3309]

In this paper, making use of the ideas of Selling [J. Reine Angew. Math. 77, 143–229 (1873)], the author gives a geometrical method for the construction of a fundamental domain in the interior of the asymptotic cone of the hyperboloid $f(x, y, z)=D$, of the group Γ of automorphs of an indefinite ternary quadratic form $f(x, y, z)$ of determinant $D>0$. The author shows that a fundamental domain H of Γ on this hyperboloid may be obtained by combining in a certain way a number of adjacent fields of Selling. Then, by drawing rays from the origin through points of H, he obtains a fundamental domain of Γ inside the asymptotic cone, or, as one may put it, a fundamental domain of Γ on every surface $f=k>0$. Each of the above fundamental domains on $f=D$ is bounded. They cover the whole surface without overlapping and are transformed into one another by the automorphs of f. Each domain has but a finite number of neighboring domains and the group Γ has but a finite number of generators. It is next shown that to every fundamental domain on $f=D$ one may assign a bounded domain (called "basic" by the author) on $f=-D$, which, although not a fundamental domain of Γ, contains at least one point equivalent to any given point on the surface. In this manner the author solves the problem of Markoff [Mém. Imp. Acad. Sci. St. Pétersbourg (8) 23, no. 7 (1909)], that is, the problem concerning the existence of "reduced"

solutions of a diophantine equation $f=m$, $m\leqq 0$. All of these results are obtained under the non-trivial restriction that f should not represent zero properly.

One should mention that in a very elegant development of the theory of quadratic forms [Abh. Math. Sem. Hansischen Univ. **13**, 209–239 (1940); these Rev. **2**, 148] Siegel obtained, without the above restriction, similar results for the automorphs of an indefinite quadratic form of index $(n, m-n)$, in any number m of variables. There a fundamental domain is constructed in a suitably chosen space H of $n(m-n)$ dimensions. It may be of interest to note that the specialization of these general results to the case of this paper permits of a geometric interpretation closely related to that given here. The author refers to another general study by Wenkov [Bull. Acad. Sci. USSR **1937**, 139–170].

A. E. Ross (St. Louis, Mo.).

Koksma, J. F. und Meulenbeld, B. Ueber die Approximation einer homogenen Linearform an die Null. Nederl. Akad. Wetensch., Proc. **44**, 62–74 (1941). [MF 4046]

Let S_n be any system of n real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ ($n\geqq 1$, fixed); and let

$$L_n=\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n-y,$$

where $(x_1, x_2, \cdots, x_n, y)$ is a lattice-point, and

$$X=\max(|x_1|, |x_2|, \cdots, |x_n|)\geqq 1.$$

L_n is said to admit the approximation $\phi(X)$ if there are infinitely many systems $(x_1, x_2, \cdots, x_n, y)$ for which

$$|\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n-y|<\phi(X).$$

It is well known that L_n admits the approximation $1/X^n$ ($n\geqq 1$). The authors show that L_n admits the approximation $1/c_nX^n$ if

$$c_n\leqq(1+1/n)^n\{1+((n-1)/(n+1))^{n+2}\}.$$

The method is geometrical. By means of a lemma the authors are able to translate to linear forms L_n the volume calculations used by H. F. Blichfeldt [Trans. Amer. Math. Soc. **15**, 227–235 (1914)] to prove the analogous result concerning the simultaneous rational approximation of S_n.

D. C. Spencer (Cambridge, Mass.).

Koksma, J. F. Ueber die Diskrepanz (mod 1) **und die ganzzahligen Lösungen gewisser Ungleichungen.** Nederl. Akad. Wetensch., Proc. **44**, 75–80 (1941). [MF 4047]

Suppose $N\geqq 1$; let U be a system of N real numbers $f(1), f(2), \cdots, f(N)$, and α, β a pair of real numbers satisfying $\alpha\leqq\beta\leqq\alpha+1$. Let $A(\alpha, \beta)$ be the number of solutions of the Diophantine inequality $\alpha\leqq f(x)<\beta$ (mod 1), and write

$$R=R(\alpha, \beta)=A(\alpha, \beta)-(\beta-\alpha)N. \tag{1}$$

The upper limit D of the numbers $|R|/N$ for all pairs (α, β), $\alpha\leqq\beta\leqq\alpha+1$, is called (after van der Corput) the discrepancy (Diskrepanz) (mod 1) of the system U. Let U^* be the system of the N^2 differences $f(x)-f(z)$ $(x, z=1, 2, \cdots, N)$, D^* the discrepancy (mod 1) of U^*. van der Corput and Pisot [references given in the author's paper] have shown that

$$D\leqq 2^{\frac{1}{2}+(1/4\epsilon)}\cdot D^{*\frac{1}{2}-\epsilon}, \qquad \epsilon>0. \tag{2}$$

If $D(N)$ is the discrepancy (mod 1) of the first N terms of the system composed of an infinite series of real numbers $f(1), f(2), \cdots$, and if $H(N)$ tends with increasing N monotonically to zero and satisfies $H(N)>D(N)$, then it follows at once from the definition of D that the Diophantine inequality

$$\alpha-H(x)<f(x)<\alpha+H(x) \pmod 1 \tag{3}$$

possesses infinitely many distinct integral solutions x. If $D^*(N)\to 0$, the van der Corput-Pisot inequality (2) implies that, for every α, $H(x)$ in (2) may be taken approximately of the order $(D^*(x))^{\frac{1}{2}}$. Stated roughly, the author shows that, for almost all α, if $H(x)$ is taken approximately of order $D^*(N)$, then (3) has an infinity of solutions.

The author's result is deduced from the following identity [first stated in terms of sums by Vinogradoff, later in its present form by van der Corput and Pisot; Nederl. Akad. Wetensch., Proc. **42**, 476–486, 554–565, 713–722 (1939); cf. these Rev. **1**, 66]:

$$\int_0^1 R^2(\alpha-t, \alpha+t)d\alpha=\int_0^{2t} R^*(-\alpha, \alpha)d\alpha, \quad 0\leqq t\leqq\tfrac{1}{2}, \tag{4}$$

where $R(\alpha, \beta)$, $R^*(\alpha, \beta)$ are the error terms defined by (1) in terms of the systems U, U^*, respectively. The author gives a simple direct proof of (4).

D. C. Spencer.

Parry, C. J. The p-adic generalization of the Thue-Siegel theorem. J. London Math. Soc. **15**, 293–305 (1940). [MF 4040]

Let ξ be an algebraic number of degree n and s a positive integer. By a result of the reviewer [Math. Z. **10**, 173–213 (1921)], the inequality $|qr^{-1}-\xi|<r^{-(s+\epsilon+n(s+1)^{-1})}$ has for any positive ϵ only a finite number of solutions in integers q and $r>0$; a corresponding statement holds in the case when approximation is made by algebraic numbers of a fixed degree instead of the rational number q/r. Another generalization of Thue's theorem was found by Mahler [Math. Ann. **107**, 691–730 (1933) and **108**, 37–55 (1933)] who considered approximations to p-adic numbers by rational numbers and obtained an estimation of the number of representations of an arbitrary integer by binary forms. The author combines these results and extends them to the case of approximation by algebraic numbers for the different possible valuations. The proofs were to have been published in Acta Arithmetica; the present note contains only an abstract of the results.

C. L. Siegel.

Bullig, G. Zur Kettenbruchtheorie im Dreidimensionalen (Z 1). Abh. Math. Sem. Hansischen Univ. **13**, 321–343 (1940). [MF 2389]

This paper is the first of a series to be designated by Z 1, Z 2, etc., in which the author studies multi-dimensional continued fractions and related problems in the theory of numbers. In the present paper, the author develops the foundation for a generalization of Minkowski's theory of continued fractions to sets M of points in three-dimensional Euclidean space R_3 which satisfy the following four axioms: (1) all points of M lie in the quadrant $x_1>0$, $x_2>0$, $x_3>0$ of R_3; (2) M has no limit point in R_3; (3) at most one point of M lies on any plane $x_i=\text{constant}$; (4) at least one point of M lies in any domain $0<x_i<c_i$, $0<x_k<c_k$, $i\neq k$, where c_i, c_k are any positive numbers. Parallelepipeda Q defined by the inequalities $0<x_i<c_i$, $i=1, 2, 3$, are considered, and extreme (in the sense of Minkowski) Q's are divided into three classes, an extreme Q of the third sort having three distinct points of M on its boundary. Let V be the sum of all Q's of the third sort, $B(V)$ the part of the boundary of V lying in the octant $x_i>0$. The author (i) studies the structure of $B(V)$, proving in particular that $B(V)$ is

mapped topologically on the plane $\xi_1+\xi_2+\xi_3=0$ by the transformation

$$(\xi_n)=\left(\ln\frac{x_1}{(x_1x_2x_3)^{\frac{1}{3}}}, \ln\frac{x_2}{(x_1x_2x_3)^{\frac{1}{3}}}, \ln\frac{x_3}{(x_1x_2x_3)^{\frac{1}{3}}}\right);$$

(ii) defines operations $\Omega(Q)$ by means of which other Q's of the third sort are derivable from a given Q of the third sort; (iii) proves that a suitable series of operations Ω leads from any Q of the third sort to any other Q of the third sort.

The result (iii) (as the author points out) is a well-known result of Minkowski when M is a lattice.

D. C. Spencer (Cambridge, Mass.).

ANALYSIS

Calculus

✶**Lohr, Erwin. Vektor- und Dyadenrechnung für Physiker und Techniker.** Walter de Gruyter & Co., Berlin, 1939. xv+411 pp. RM 18.

The notation used in this book is that of Gibbs and of the author's teacher, G. Jaumann. Mathematical rigor has not been sought in presenting the basic concepts and definitions; instead, geometrical and physical significances are emphasized. The scope of the subject matter is exceptionally ambitious, and the book is characterized by its attention to a great number of different mathematical and physical concepts.

The text is divided into three main parts: (I) the arithmetic and algebra of hyper-numbers (extensiver Grössen), (II) the analysis of hyper-numbers, and (III) physical applications. In the first part, vectors, dyadics, triadics, tetradics, bivectors, bidyadics, etc., are introduced, the principal algebraic operations involving them are defined and discussed, and their most important properties are shown. A section is devoted to "eigenvalue problems of dyadics" and another to the invariants of dyadics, Cayley-Hamilton's equation and "the dyadic as a deformation-dyadic." In the second part, the differential and integral operations are introduced, including dyadic differentiation and hyper-derivatives of higher orders and ranks. Line, surface and volume integrals and their relationships are treated, and several sections are devoted to source and vortex fields and their connections with vector and dyadic field functions. Scalar and vector potentials are defined and are evaluated for point, line and sheet distributions of sources and vortices, double layers, etc. In the third part, applications to mechanics, geometry, elasticity, hydrodynamics, electromagnetics, optics and quantum mechanics are illustrated. Each of these sections consists of about twenty pages.

The word "tensor" is used in this text only in the sense of a "pure tensor" (e.g., a pure strain), although the author points out the more general meaning of the word in a footnote. The summation convention is not employed in any of the formulae. *W. R. Sears* (Pasadena, Calif.).

Lotze, A. Die elementaren Differentialoperationen in der Grassmannschen Vektoranalysis. Jber. Deutsch. Math. Verein. 50, 79–91 (1940). [MF 2776]

The original Grassmann vector analysis of three-dimensional euclidean affine space deals with vectors, bivectors and trivectors. Bivectors are introduced as the outer product of two vectors; trivectors as the outer product of three vectors. The author defines the concepts: (1) outer product of bivectors (denoted by the symbol ·); "Ergänzungsstrich" operating on a vector or bivector (denoted by the symbol |). The properties of these symbols are then discussed. [Reviewer's note: these operators are related to Schouten's three-vector.] In the second section of the paper, the author shows that these operations furnish a systematic scheme for determining complicated vector identities. In conclusion, generalizations of Stoke's and Gauss' theorems are proven.

N. Coburn (Austin, Tex.).

Bloch, André. Sur les systèmes d'aires planes orientées dans l'espace. C. R. Acad. Sci. Paris **210**, 728–729 (1940). [MF 3918]

An outline of a theory of oriented areas, each sliding in its plane, analogous to the theory of force vectors, each sliding in its line. *P. Franklin* (Cambridge, Mass.).

Calcagno, Horacio E. Essay on an Archimedean system. Revista Union Mat. Argentina 7, 12–17 (1940). (Spanish) [MF 3855]

The author considers the ordered field of complex numbers obtained by defining $x<y$ whenever (1) $|x|<|y|$, (2) $\arg x<\arg y$ if $|x|=|y|$, and investigates the complex interval $\{x, y\}$ determined by the set of complex numbers z such that $x\leqq z\leqq y$. This interval is measured by a tripartite number $(A; l_x, l_y)$ of three components, where A is the area of the annular region given by $|x|\leqq|z|\leqq|y|$, $l_x=(2\pi-\arg x)\cdot|x|$, and $l_y=|y|\cdot\arg y$. Equality of two such tripartite numbers and their sum and product by a scalar are defined in the usual manner. If $a<b$, it is shown how the complex number x is determined for which $\{a, b\}=\{c, x\}$, c a given complex number.

L. M. Blumenthal (Columbia, Mo.).

Tambs Lyche, R. On Rolle's theorem. Norsk Mat. Tidsskr. **22**, 105–109 (1940). (Norwegian) [MF 3758]

Raikov, D. Sur les différences et sur les dérivées. Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 379–384 (1940). (Russian. French summary) [MF 2799]

A simple proof is given of the following theorem: If $f(x)$ is continuous in $[a, b]$ and if there exists a sequence $h_k\to 0$ such that

$$|\Delta^n_{h_k}f(x)|\leqq C|h_k|^n$$

for each $x\epsilon[a, b]$ and each h_k such that $x+nh_k\epsilon[a, b]$, where C is independent of x and h_k, then $f(x)$ has an $(n-1)$st derivative in $[a, b]$ satisfying for each $x, y\epsilon[a, b]$ the Lipschitz property

$$|f^{(n-1)}(y)-f^{(n-1)}(x)|\leqq C|y-x|.$$

The proof is based on a generalization of Rolle's Theorem due to S. Bernstein. The author had previously proven this theorem in another manner [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 652–655 (1939); these Rev. **1**, 333].

J. V. Wehausen (Columbia, Mo.).

Khudekov, I. N. On a formal property of iterated functions. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **6**, 115–118 (1939). (Russian) [MF 3295]

It is known that if a sequence of iterations $f(z), ff(z), fff(z), \cdots$ of an analytic function converges its limit must

be a root z_0 of the equation $f(z)=z$ if only $|f'(z_0)|\leqq 1$. A formal proof of this theorem is given by using power series. Assumptions are not clearly specified. *M. Kac.*

Żyliński, E. Sur la méthode des multiplicateurs de Lagrange. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 68–71. [MF 3504]

A proof of the validity of the method of Lagrange's multipliers for the determination of the stationary points of a function $f(x_1, \cdots, x_m; y_1, \cdots, y_n)$ subject to the conditions $\varphi_i(x_1, \cdots, x_m; y_1, \cdots, y_n)=0$, $i=1, 2, \cdots, n$.

J. W. Calkin (Chicago, Ill.).

Bonferroni, C. E. Di uno speciale determinante formato con determinanti di Gram e di Landsberg. Boll. Un. Mat. Ital. (2) **2**, 115–121 (1940). [MF 2970]

Let $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ be $2n$ real functions of the real point P, defined and L-integrable on a certain k dimensional domain ω. Let us form the determinant (already considered by Landsberg)

$$G_{uv}=\|\int u_i v_k dP\|.$$

Let us have m sets of n functions $u_i, v_i, \cdots, w_i$ $(i=1, \cdots, n)$ and let us form the new determinant

$$H_{uv\cdots w}=\begin{vmatrix} G_{uu} & G_{uv} & \cdots & G_{uw} \\ \cdot & \cdot & \cdot & \cdot \\ G_{wu} & G_{wv} & \cdots & G_{ww} \end{vmatrix}.$$

The author proves that $H_{uv\cdots w}$ is the Gram determinant of certain m functions related to the given functions, whence it follows that $H_{uv\cdots w}\geqq 0$. It is shown that $G_{uv}=0$ and $H_{uv}=0$ are necessary and sufficient conditions for the existence of a linear combination of the u_i which is orthogonal to all the v_i, and for the linear equivalence of the u_i and v_i, respectively. *A. González Domínguez* (Buenos Aires).

Jacobsthal, Ernst. Über die eineindeutige Abbildung zweier Bereiche auf einander bei nichtverschwindender Funktionaldeterminante. Norske Vid. Selsk. Forh. **13**, 123–126 (1940). [MF 3872]

Jacobsthal, Ernst. Über den Mittelwertsatz der Integralrechnung. Norske Vid. Selsk. Forh. **13**, 27–29 (1940). [MF 3867]

Brown, A. B. On transformation of multiple integrals. Amer. Math. Monthly **48**, 29–33 (1941). [MF 3720]

The proof of the formula for transformation of a multiple Riemann integral under a nonsingular change of variables on a region R in a (u)-space in n variables to an (x)-space depends on the formula for the volume V of the image of R. The author assumes appropriate stated conditions imposed upon the region, the derivatives and the transformation, including in particular that the Jacobian does not vanish in the region considered. The "standard proof" for the volume formula includes the use of a surface integral where the surface is represented parametrically. The author adopts the method based on considering the image of an n-dimensional rectangular solid. The linear approximation to the given transformation is obtained for each fixed point (u^0) of R. The set of all points in the (u, u^0)-space for which the linear approximation is sufficiently close is considered, and the error in volume, for boundary points, is estimated. By properly passing to the limit, a short and satisfying proof is obtained. *A. A. Bennett* (Providence, R. I.).

Higgins, Thomas James. Note on an integral of Bierens de Haan. Bull. Amer. Math. Soc. **47**, 286–287 (1941). [MF 4177]

The author points out an error in a rule given in Table 371 of Bierens de Haan's Nouvelles Tables d'Intégrales Définies for the evaluation of:

$$\int_0^\infty e^{-p^2x}x^{-n-1}\sin q_0x\sin q_1x\cdots\sin q_nx\,dx.$$

He states without proof a corrected form of the rule. The rule could be proved, or the integral evaluated, by replacing the sines by expressions in complex exponentials, integrating by parts repeatedly, and using the known results [cf., for example, Franklin, A Treatise on Advanced Calculus, Ex. 30, 31, p. 420; 35, 36, p. 421]:

$$\int_0^\infty e^{-p^2x}x^{-1}\sin kx\,dx=\tan^{-1}(k/p)$$

and

$$\int_0^\infty e^{-p^2x}x^{-1}(\cos kx-\cos cx)dx=\tfrac{1}{2}\log((c^2+p^2)/(k^2+p^2)).$$

P. Franklin (Cambridge, Mass.).

Oakley, C. O. Equations of polygonal configurations. Amer. Math. Monthly **47**, 621–627 (1940). [MF 3263]

This paper discusses single equations which represent broken lines or areas bounded by polygons. The functions signum x and absolute value of x are the main tools used in the development. *P. Franklin* (Cambridge, Mass.).

Maeda, Jusaku. A remark concerning plane curves. Sci. Rep. Tôhoku Imp. Univ., Ser. 1. **28**, 334–349 (1940). [MF 1711]

The author studies the cubic which is the locus of the foci of those conics which have at least four-point contact with a given curve (M) at a given point M. He considers some other similar curves and loci connected with the behavior of (M) at M. *P. Scherk* (New Haven, Conn.).

Sanguineti, Jeronimo. Practical considerations on the shape of the cubatrix and the strophoide. An. Soc. Ci. Argentina **129**, 32–42 (1940). (Spanish) [MF 3774]

The author obtains constructions and equations for the curve having the polar equation $\rho^2\cos\theta=1$ and some other curves. [Cf. also these Rev. **1**, 165.] *W. Feller.*

Theory of Sets, Theory of Functions of Real Variables

Sherman, Seymour. Some new properties of transfinite ordinals. Bull. Amer. Math. Soc. **47**, 111–116 (1941). [MF 3821]

In the Cantor normal form for transfinite ordinals (as established by Sierpinski) each ordinal greater than 0 is expressed as a finite sum of monomial terms in descending powers of ω (with finite right coefficients). Using this form the author investigates further the nature of left and of right factors of transfinite ordinals. In particular for ordinals A, B, C, it is shown that $(A+B)C\leqq AC+BC$. He obtains also necessary and sufficient conditions for the equality to hold. *A. A. Bennett* (Providence, R. I.).

Livenson, E. On the realization of Boolean algebras by algebras of sets. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 309–312 (1940). (English. Russian summary) [MF 2792]

Stone's theorem, that every Boolean algebra is isomorphic to a field of sets, is given another proof. The transfinite induction involved is correlated explicitly with ordinals; the word "ideal" is suppressed. *G. Birkhoff.*

Stone, M. H. Characteristic functions of families of sets. Duke Math. J. 7, 453–457 (1940). [MF 3413]

The notion of the characteristic function of a sequence of sets which was introduced by Kuratowski and developed further by Szpilrajn [Fund. Math. 26, 302–326 (1936)] is defined algebraically in terms of the concepts used in the author's discussion of Boolean rings. An application is made by proving that any T_0 space of infinite character c is a bi-univocal continuous image of a subset of a totally-disconnected bicompact Hausdorff space of character c.

N. Dunford (New Haven, Conn.).

Arsenin, B. Sur les projections de certains ensembles mesurables B. C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 107–109 (1940). [MF 3212]

Soit E un ensemble mesurable B situé sur le plan xOy. Soit ensuite P l'ensemble des points de E situés sur la droite $x=x_0$ dont chacune coupe l'ensemble E en un ensemble du type F_σ non vide. L'auteur démontre que la projection sur l'axe Ox de P est un ensemble CA. Ce résultat donne une solution d'un problème de Szpilrajn [Fund. Math. 24, 324 (1935)]: La projection d'un ensemble E plan mesurable B, tel que chaque droite $x=x_0$ coupe E en un ensemble du type F_σ, est un ensemble mesurable B. Cette proposition généralise un théorème fondamental de M. Lusin sur les points d'unicité. La démonstration est basée sur une méthode de P. Novikoff et sur un lemme intéressant: Soit M un ensemble du type F_σ situé sur l'axe Ox; soit ensuite N un ensemble G_δ uniforme par rapport à l'axe Ox et dont la projection sur l'axe Ox coïncide avec M; on a $N=s_1\cdot s_2\cdots s_n\cdots$, où s_n est une somme de rectangles (sans leurs frontières) de rang n, mutuellement disjoints et dont les côtés sont parallèles aux axes x et y; $s_n\subset s_{n-1}$. Alors, il existe parmi eux un rectangle tel que la fermeture de l'ensemble que l'on obtient en projetant sur l'axe Ox la partie de N contenue dans ce rectangle appartient à M. Ces résultats et ce lemme sont retrouvés par Kunugui [Proc. Imp. Acad. Tokyo 16, 73–78 (1940); these Rev. 1, 302].

K. Kunugui (Sapporo).

Keldych, Ludmila. Sur les ensembles homogènes mesurables B. C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 523–525 (1940). [MF 3558]

Keldych, Ludmila. Démonstration directe du théorème sur l'appartenance d'un élément canonique E_α à la classe α et exemples arithmétiques d'ensembles mesurables B de classes supérieures. C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 675–677 (1940). [MF 3612]

The author continues her previous work on canonical elements of the higher Baire classes [Rec. Math. [Mat. Sbornik] 41, 187–220 (1934)] using the representation in terms of perfect sets and the operation A [Bull. Acad. Sci. URSS 2, 221–248 (1938)]. The methods of the first note are improved in the second so that the classes of these canonical elements are determined without the use of the existence of Baire sets of all classes. Effective examples are given of canonical elements of classes $4, 5, \cdots, \omega$, where the sets may be defined through rather complex number-theoretical properties of continued-fraction expansions of irrational numbers. The method will permit the effective construction of a canonical element of class α, if α is effectively given in the sense that a fundamental sequence is chosen for every limit ordinal $\beta\leqq\alpha$. *J. W. Tukey* (Princeton, N. J.).

Sokolin, A. Concerning a problem of Radó. C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 871–872 (1940). [MF 3582]

This paper gives a partial answer to the following question, stated by the reviewer in Fund. Math. 11 (1928). In the xy plane, let there be given a finite system S of squares, with sides parallel to the axes. Let A denote the point-set covered by these squares. We want to know whether it is always possible to discard some of the squares of S in such a way that (1) the remaining squares have no common interior points, and (2) the sum of the areas of the remaining squares is at least one-fourth of the measure of A. The author gives an affirmative answer in the special case when the squares of S have the same side-length. The proof is based on a theorem of Blichfeldt on square lattices.

T. Radó (Columbus, Ohio).

Sierpiński, W. Sur l'opération $\overline{\lim}_{y\to+\infty}\Phi(x,y)$. Pont. Acad. Sci. Acta 4, 203–204 (1940). [MF 3961]

In connection with an example due to Lusin [Leçons sur les ensembles analytiques et leurs applications, 1928, p. 318] of a function $\Phi(x,y)$ of (Baire) class 2 such that $f(x)=\overline{\lim}_{y\to\infty}\Phi(x,y)$ is not mesurable B, the author states: If Φ is of class not greater than 1, $f(x)$ is of class not greater than 3. It is also stated that, if, in the class of Baire functions $\Phi(x_1,\cdots,x_n)$, the operations $\lim_k\Phi_k$ and $\overline{\lim}_{x_i\to\infty}\Phi$ are applied a finite number of times in any order, the resulting functions are measurable L. *L. W. Cohen.*

Sierpiński, W. Sur une propriété des ensembles ordonnés. Pont. Acad. Sci. Acta 4, 207–208 (1940). [MF 3963]

The author announces the theorem: If ν is an ordinal number, every ordered set of power $\aleph_\nu$ is similar to a set of transfinite sequences of type ω_ν formed from the digits 0, 1 and ordered by the principle of first differences; the ordinal ω_ν can not be replaced by any smaller one. A similar result was given by Hausdorff [Grundzüge der Mengenlehre, 1927, p. 182], in which three digits are used.

L. W. Cohen (Lexington, Ky.).

Sierpiński, W. Sur les bases dénombrables des familles de fonctions. Pont. Acad. Sci. Acta 4, 211–212 (1940). [MF 3962]

A family F of functions of a real variable is said to have a denumerable basis $f_n(x)$, not necessarily in F, if $f\varepsilon F$ is the limit of a subsequence of f_n. The author denotes this property by $F\varepsilon B$ and states the following theorems: (1) If $F\varepsilon B$ and F_ρ, F_h, F_u are, respectively, the families of limits of non-decreasing, non-increasing, uniformly convergent sequences $f_n\varepsilon F$, then $F_\rho F_h\varepsilon B$, $F_u\varepsilon B$. (2) If the family ul of W. H. Young is in B, then every Baire class α, $\alpha<\Omega$, is in B. (3) $F\varepsilon B$ if and only if there exists $\phi(x)$ such that every $f\varepsilon F$ is of the form $g(\phi(x))$, where g is the limit of a sequence of polynomials on $E\{y;y=\phi(x)\}$. (4) If $F\varepsilon B$, there is a $\phi(x)$ such that the family of $p(\phi(x))$, p being a polynomial with rational coefficients, is a basis for F. The details of this and the preceding two notes should have appeared in Fundamenta Mathematicae. *L. W. Cohen* (Lexington, Ky.).

Radó, T. and Reichelderfer, P. A theory of absolutely continuous transformations in the plane. Trans. Amer. Math. Soc. 49, 258–307 (1941). [MF 3924]

The authors consider bounded continuous transformations $T: z=t(w)$ or $x=x(u,v)$, $y=y(u,v)$ defined on a bounded domain D in the $w=u+iv$ plane and introduce the following notations and terminology: If E is a set in D, $T(E)$ denotes its transform under T and if $\bar{E}$ is a set in the z plane, $T^{-1}(\bar{E})$ consists of all points w of D for which $T(w)\epsilon\bar{E}$. If z is any point, $N(z, T, E)$ denotes the number of distinct points in $E\cdot T^{-1}(z)$. If $\mathfrak{R}$ is any (finitely connected) Jordan region in D, $\mu(z, T, \mathfrak{R})$ is the topological index of z with respect to $T(\mathfrak{R}^*)$ ($\mathfrak{R}^*$=boundary of $\mathfrak{R}$) unless z is on $T(\mathfrak{R}^*)$, when $\mu=0$. Let z be a point and let $o(z, T)$ be a component of $T^{-1}(z)$ which is a continuum interior to D; if every open set containing $o(z,T)$ contains a Jordan region $\mathfrak{R}\supset o(z,T)$ such that $\mu(z,T,\mathfrak{R})\neq 0$, $o(z,T)$ is said to be an essential maximal model continuum of z. The fundamental set $\mathcal{E}$ consists of all points w_0 each of which is itself an essential maximal model continuum of $t(w_0)$. The set $\mathfrak{N}$ consists of all w_0 in $\mathcal{E}$, each of which is an isolated point of $\mathcal{E}\cdot T^{-1}[t(w_0)]$. If $w_0\epsilon\mathfrak{N}$, $j(w_0, T)$ is defined equal to $\mu[t(w_0), T, \mathfrak{R}]$ for any Jordan $\mathfrak{R}$ containing w_0 but no other point of $\mathcal{E}\cdot T^{-1}[t(w_0)]$; otherwise $j(w_0,T)=0$. It is shown that $j(w_0,T)=\pm 1$ for all except a denumerable number of points of $\mathfrak{N}$. If Δ is any subdomain of D and z is any point, $\nu(z,T,\Delta)=\sum_w j(w,T)$ for all w in $\mathcal{E}\cdot\Delta\cdot T^{-1}(z)$, provided $N(z,T,\mathfrak{N}\cdot\Delta)\neq\infty$, in which case $\nu=0$.

Next, let $\mathfrak{B}$ be a measurable set in the w plane which is such that $T(R\cdot\mathfrak{B})$ is measurable in the z plane for every open cell R in D. T is said to be B.V.$\mathfrak{B}$. or A.C.$\mathfrak{B}$. in D according as the cell function $G(R)=\text{meas}[T(R\cdot\mathfrak{B})]$ is B.V. or A.C. on D. The theorems proved by Banach for the case $\mathfrak{B}=D$ [Fund. Math. 7, 225–236 (1925)] are all proved for any $\mathfrak{B}$. In particular, the Banach derivative $D(w,T,\mathfrak{B})$ of $G(R)$ exists almost everywhere and is summable if T is B.V.$\mathfrak{B}$. It is next shown that the above set $\mathcal{E}$ is an admissible set $\mathfrak{B}$ and the authors define the class $K_1(D)$ as all T which are A.C.$\mathcal{E}$. on D. If $T\epsilon K_1(D)$ and $H(z)$ is any finite valued measurable function, the formula

$$\text{(A)}\quad \iint_D H[t(w)]D(w,T,\mathcal{E})dudv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(z)\cdot N(z,T,D\cdot\mathcal{E})dxdy$$

holds in the sense that both sides exist simultaneously and also

$$\text{(B)}\quad \iint_D H[t(w)]\cdot D(w,T,\mathcal{E})\cdot j(w,T)dudv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(z)\cdot\nu(z,T,D)dxdy$$

holds if the integral on the left exists. The class $K_2(D)$ consists of all T such that $N(z,T,\mathcal{E})$ is equal for almost all z to the number of essential maximal model continua of z in D. The class $K_3(D)$ consists of these T in $K_2(D)$ for each of which the ordinary Jacobian $J(w,T)=x_u y_v - x_v y_u$ is defined almost everywhere. For such T, the quantities $D(w,T,\mathcal{E})$ and $D(w,T,\mathcal{E})\cdot j(w,T)$ may be replaced in formulas (A) and (B) by $|J(w,T)|$ and $J(w,T)$, respectively. These results include all previously known results on this subject. Certain closure theorems are proved for the classes $K_2(D)$ and $K_3(D)$. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Misra, R. D. On a new non-differentiable function. Quart. J. Math., Oxford Ser. 11, 225–228 (1940). [MF 3784]

In a previous paper [Proc. Benares Math. Soc. 16, 13–34 (1934)] the author has set up an explicit parametric representation $x=f_1(t)$, $y=f_2(t)$ $(0\leqq t\leqq 1)$ for the space-filling curve of Hilbert [Math. Annalen 38 (1891)]. It is now shown that $f_1(t)$ is nowhere differentiable and also devoid of cusps. *I. J. Schoenberg* (Waterville, Me.).

Cameron, R. H. and Martin, W. T. An unsymmetric Fubini theorem. Bull. Amer. Math. Soc. 47, 121–125 (1941). [MF 3824]

Let $k(x)$ be of bounded variation on every finite interval; let $p(x,u)$ be B-measurable and suppose that the variation $V(x,u)$ of $p(x,v)$ on $0<v\leqq u$ is finite for all x except at most a set of $k(x)$-measure zero. Assume that $\int_{-\infty}^{+\infty} V(x,u)|dk(x)|$ exists for all u. Then for any measurable $s(u)$

$$\int_{-\infty}^{+\infty}|s(u)|d_u\int_{-\infty}^{+\infty}V(x,u)|dk(x)| = \int_{-\infty}^{+\infty}|dk(x)|\int_{-\infty}^{+\infty}|s(u)|\cdot|d_u p(x,u)|,$$

in the sense that the finiteness of either integral implies that of the other; moreover the equality holds also if $V(x,u)$ be replaced by $p(x,u)$ and the absolute values be omitted. A similar theorem occurs as a lemma in a recent paper of the reviewer [Trans. Amer. Math. Soc. 48, 488–515 (1940); these Rev. 2, 101]. *W. Feller* (Providence, R. I.).

Radó, T. and Reichelderfer, P. Note on an inequality of Steiner. Bull. Amer. Math. Soc. 47, 102–108 (1941). [MF 3819]

Let $f(x,y)$ be a continuous function defined on the square $Q: 0\leqq x, y\leqq 1$, and let $L(f)$ denote the Lebesgue area of the surface $z=f(x,y)$. If $L(f)$ is finite, it is known that f_x and f_y exist almost everywhere and are summable and that

$$\iint_Q (1+f_x^2+f_y^2)dxdy\leqq L(f).$$

If $\{f_n(x,y)\}$ converges uniformly to $f(x,y)$ on Q, then $L(f)\leqq\liminf_{n\to\infty}L(f_n)$. McShane [Ann. of Math. (2) 33, 125–138 (1932)] has proved that, if f_n tends uniformly to f on Q and if $L(f_n)\to L(f)$, then

$$\varphi_n(x,y)=[(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{\frac{1}{2}}$$

tends to zero in measure but not necessarily in the mean of order 1. The authors prove that $\varphi_n(x,y)$ does tend to zero in the mean of any order λ with $0<\lambda<1$. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Fleddermann, Harry T. Equality between measure functions. Bol. Mat. 13, 295–297 (1940). (Spanish) [MF 3445]

The author replaces Carathéodory linear measure by Gross linear measure in Besicovitch's definition of the linear density of a plane set, and obtains immediately from known theorems the result that, if either density is equal to unity at every point of the set, then the other is equal to unity at every point of the set with the possible exception of a subset of Carathéodory linear measure zero. [The author's use of the phrase "almost everywhere," without specification of the measure function involved, is somewhat confusing.] Hence the author deduces from Randolph's covering theorem [Ann. of Math. (2) 40, 299–308 (1939)] that

a set having either density equal to unity at each point has its Carathéodory and Gross linear measures equal; however, his proof is not valid if the set has infinite Carathéodory linear measure (although the theorem remains true even in this case). *R. P. Boas, Jr.*

Fan, S. C. Integration with respect to an upper measure function. Amer. J. Math. 63, 319–338 (1941). [MF 4153]

This paper gives a comprehensive theory of integration for arbitrary functions over arbitrary sets E. It is worked out for Lebesgue exterior measure mE, but holds also for more general measure functions. Let $\bar{\mu}(y)=\bar{m}E(f<y)$, and let $m_f-\epsilon=y_0<y_1<\cdots<y_n=M_f+\epsilon$ be a subdivision of the range of f, m_f, M_f the bounds of f. If as $y_i-y_{i-1}\to 0$ the sum

$$(1)\qquad \sum_{i=1}^{n}\eta_i[\bar{\mu}(y_i)-\bar{\mu}(y_{i-1})],\qquad y_{i-1}\leqq\eta_i\leqq y_i,$$

tends to a limit, this limit is the integral $\int_E fd\bar{\mu}$ of f over E. It is a Stieltjes integral of y with respect to the non-decreasing function $\bar{\mu}(y)$. This Stieltjes integral exists for every bounded function, and if f is summable it represents the Lebesgue integral of f.

If in (1) $\bar{\mu}^*(y)=\bar{m}E(f>y)$ and $\bar{\mu}^*(y_{i-1})-\bar{\mu}^*(y_i)$ are used, an integral $\int_E fd\bar{\mu}^*$ is obtained. The possibility of similar integrals with interior measure replacing exterior measure is pointed out. If $\int_E fd\bar{\mu}=\int_E fd\bar{\mu}^*$ the function f is said to be μ-integrable. The function f is relatively measurable if the sets $E_1(f<a)$, $E_2(f\geqq a)$ are separated. (E_1, E_2 are separated if for every $\epsilon>0$ there exists open $A\supset E_1$ and open $B\supset E_2$ with $m(AB)<\epsilon$.) A necessary and sufficient condition that f be μ-integrable over E is that f be relatively measurable over E. These definitions are extended to unbounded functions, and the properties of the integrals are studied. The relations among these integrals, the Hildebrandt integral and the Pierpont integral are determined.

R. L. Jeffery (Wolfville, N. S.).

Izumi, Shin-ichi and Nakamura, Masahiko. An abstract integral, III. Proc. Imp. Acad. Tokyo 16, 518–523 (1940). [MF 3893]

[The first two communications appeared in the same volume, 21–25, 87–89; cf. these Rev. 1, 239, 305.] The underlying idea of the first part of this note is that in the general integral of Daniell [Ann. of Math. (2) 19, 280 ff. (1918)] the rôle of real functions on a general range can be taken by a restricted or σ-lattice. Then an R integral is essentially a linear positive functional f defined on a subset of the lattice subject to the continuity condition that $\lim x_n=0$, $|x_n|\leqq y$ for all n, implies $\lim f(x_n)=0$. For an L-integral, F on a subset L of the lattice, this continuity condition is replaced by two: (a) $\lim z_n=z$, $|z|\leqq y$, implies z belongs to L and $\lim F(z_n)=F(z)$; and (b) $z_n\leqq z_{n+1}$, $\lim z_n=z$, $\lim F(z_n)$ finite implies z belongs to L and $\lim F(z_n)=F(z)$. By introducing a multiplication idea into the lattice and postulating a Schwarz inequality, a Riesz-Fischer theorem becomes possible. *T. H. Hildebrandt* (Ann Arbor, Mich.).

MacNeille, H. M. A unified theory of integration. Proc. Nat. Acad. Sci. U. S. A. 27, 71–76 (1941). [MF 3657]

A series $\sum u_n(x)$ of real step functions is said to be absolutely convergent in case $\sum\int|u_n(x)|dx<\infty$. For such series $s(x)=\sum u_n(x)$ exists almost everywhere and such functions are called summable. The integral of a summable function is defined by $\int s(x)dx=\sum\int u_n(x)dx$. The facts that the space of summable functions is complete and that step functions are dense are immediate consequences of the definition. The same idea was used by the reviewer [Trans. Amer. Math. Soc. 37, 441–453 (1935)] in connection with sequences instead of series. The Denjoy integral is defined by transfinite induction and it is stated that both definitions apply equally well to the case of functions defined over a normed Boolean ring and with values in a Banach space.

N. Dunford (New Haven, Conn.).

GEOMETRY

Petrini, H. Précis d'un exposé des principes de la géométrie. Ark. Mat. Astr. Fys. 27 A, no. 10, 17 pp. (1940). [MF 3491]

This is an intuitive-logical treatment of the foundations of Euclidean geometry that recalls to the reviewer Cavalieri's method of indivisibles. Seeking a geometric concept that is given immediately by experience, the author finds it in the notion of surface which he uses as primitive. If a surface is so thin that its width may be disregarded it is called a line (curve), while a point arises from a line by abstracting the notion of length. It is supposed that the surface is bounded by a closed curve. As in Cavalieri, a surface (curve) can be considered as generated by a curve (point). Straight lines are curves which have a single point in common and cut when prolonged beyond this point. The axioms include one of exclusion for the comparison of limited straight lines (that is, the notions of "greater than," "equal to" and "less than" are pairwise mutually exclusive), an axiom of circles (two circles cannot meet in more than one point on a side of the straight line joining their centers), an axiom of homogeneity which allows displacements of figures without altering distances, and the parallel axiom. A plane is defined as a homogeneous surface upon which the parallel axiom is valid. The theory of angles requires no additional postulates. *L. M. Blumenthal* (Columbia, Mo.)

Busemann, Herbert. On Leibniz's definition of planes. Amer. J. Math. 63, 101–111 (1941). [MF 3633]

The author seeks to modify Leibniz's definition of a plane $m(A, B)$ as "the locus of points equidistant from two distinct points A, B", so that it will be applicable to a metric space S. Assuming that S is convex, externally convex and finitely compact, and moreover, that for any two points X, Y in $m(A, B)$ every point of the line XY is contained in $m(A, B)$, it is possible to conclude that S is congruent to a Euclidean or hyperbolic space of some finite dimension. The argument is based upon a consideration of the group of motions which leaves a point fixed. *G. de B. Robinson.*

Kubota, Tadahiko. Eine Begründung der elementaren Geometrie. III. Fünfter Abschnitt. Begründung der Anordnungstheorie in der projektiven Geometrie. Tôhoku Math. J. 47, 294–303 (1940). [MF 4029]

Diese Note ist für den pädagogischen Zweck verfasst um die Theorie der Anordnung der Punkte auf der Geraden in der projektiven Geometrie streng und systematisch aufzubauen. *Extract from the paper.*

Rachevsky, P. Sur une géométrie projective avec de nouveaux axiomes de configuration. Rec. Math. [Mat. Sbornik] N. S. **8**(**50**), 183–204 (1940). (French. Russian summary) [MF 3459]

Consider two classes "points" and "lines" of abstract elements, together with a primitive relation "on" such that (1) each pair of distinct points (lines) is on one and only one line (point) and (2) each line (point) is on at least three distinct points (lines). The author studies the projective plane geometries obtained by adjoining to the above system one of the two axioms of configuration: Axiom (7). The three diagonal points of a complete quadrangle are on a line. Axiom (8). If the two ordered triples A, B, C and A', B', C' on lines l and l', respectively, are perspective with respect to a point S, then A, B, C and B', C', A' are perspective with respect to a point S'.

It is shown that if each line (point) is on at least five points (lines) then Axiom (8) implies Axiom (7). It follows from a previous paper of the author's [Rec. Math. [Mat. Sbornik] **8** (**50**), 107–120 (1940); these Rev. **2**, 135] that the geometries obtained are finite, axioms of order and continuity being necessarily absent. If the theorem of Desargues is valid in geometry (7) then the fundamental field in terms of which the analytic geometry is based has characteristic 2. An interesting proof shows that there are exactly two kinds of geometries (8): one kind given by the configuration (13, 13) of thirteen lines and points, four points (lines) on each line (point), in which Axiom (7) is not valid, and the other kind formed by the configuration (21, 21) with each point (line) on exactly five lines (points). Their fundamental fields consist, respectively, of three and four elements; for example, (1) the field of residues modulo 3 and (2) the field of characteristic 2 with elements 0, 1, r, s with the addition and multiplication tables: $r+s=1$, $r+1=s$, $s+1=r$, $1+1=r+r=s+s=0$; $rs=sr=1$, $r^2=s$, $s^2=r$.

L. M. Blumenthal (Columbia, Mo.).

Barbilian, D. Zur Axiomatik der projektiven ebenen Ringgeometrien. I. Jber. Deutsch. Math. Verein. **50**, 179–229 (1940).

The axiomatic foundation of projective geometry falls into two parts, the first of which deals with questions of existence and incidence while the second deals with the introduction of a coordinate system and field properties. The two parts are correlated by means of von Staudt's constructions for the "sum" and "product" of two points on a line. In this paper the author investigates the conditions which must be imposed upon an arbitrary ring $\mathfrak{O}$ in order that the corresponding geometry, defined analytically, may have useful and interesting properties. He confines his attention to a space of two dimensions.

If the equation $xa=b$ is soluble for all b in $\mathfrak{O}$, then a is said to be left-regular, otherwise left-singular. Similarly, a is right-regular or right-singular.

In § 1 it is shown that the ring $\mathfrak{O}$ must have a unity element (axiom E). After introducing cogredience, contragredience and linear transformation it is possible to define point and line and to give an implicit criterion of incidence. For this to be of geometrical significance every singular element of $\mathfrak{O}$ must be two-sided singular (axiom Z). The criteria of incidence may now be stated explicitly, with the help of the so-called "modular group" of linear transformations and transitivity properties proved. In order to identify modular with projective transformations in the ordinary sense it is necessary further to restrict the ring $\mathfrak{O}$. A sufficient restriction is that if $u_1^0\xi_1+u_2^0\xi_2$ is non-singular then $\xi_1+u_2\xi_2$ and $u_1\xi_1+\xi_2$ are also non-singular. Various examples of rings having this property are given, in particular an algebra over a given field. § 2 is devoted to proving the existence of the various geometrical elements and their incidence relations based on this algebraic foundation. After the definition of projection and of the degenerate semi-projection at the beginning of § 3, the author defines the addition of points on a line by means of the Desargues (von Staudt) construction. As in the usual theory the "sum" point of the quadrangular set is uniquely determined by the other five points. The product point is similarly treated and the relations are shown to be invariant under projection. The method of proof is that originated by Möbius.

G. de B. Robinson (Toronto, Ont.).

Toepken, Heinrich. Über den Höhensatz in der absoluten Geometrie. Deutsche Math. **5**, 395–401 (1941). [MF 4251]

The author proves the concurrency of the three altitudes of a triangle using only those axioms of Hilbert's first three groups (connection, order, congruence) which refer to the plane. These axioms were the starting point of the investigations of Hjelmslev, who constructed a basis for plane (absolute) geometry in which no postulates of continuity or parallelism occur [Math. Ann. **64**, 449–474 (1907)]. The author states that it was asserted [Danske Vid. Selsk. Math.-Fys. Medd. **8**, no. 11, p. 10 (1929); not available to the reviewer] that no system other than Hjelmslev's existed [1929] in terms of which the "altitude theorem" could be proved without recourse to space considerations, continuity or a theory of parallels. If this be so, the present paper demonstrates the invalidity of this remark. The proof of the theorem makes use of two lemmas for which reference is made to Hessenberg [S.-B. Berlin Math. Ges. **4**, 69 (1905)]. *L. M. Blumenthal* (Columbia, Mo.).

de Kerékjártó, Béla. Sur la géométrie hyperbolique plane. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **59**, 19–61 (1940). (Hungarian. French summary) [MF 4445]

de Kerékjártó, B. Nouvelle méthode d'édifier la géométrie plane de Bolyai et de Lobatchefski. Comment. Math. Helv. **13**, 11–48 (1940). [MF 3986]

Selecting postulates that do not differ essentially from those of Hilbert [Grundlagen der Geometrie, 7te Aufl., 1930, Anhang III] the author develops the nature of the geometry defined in a novel fashion. From the elements of the geometry defined by the postulates a model is constructed which is proved to be Euclidean, and it is shown that the original geometry appears in this model in the form of Poincaré's realization of hyperbolic geometry which utilizes the Euclidean upper half-plane. This fresh treatment of an old problem shows incidentally that the Poincaré model is equivalent to any realization of the postulates, and establishes the existence of Euclidean geometry from that of hyperbolic geometry. *L. M. Blumenthal.*

Izumi, Shin-ichi. Lattice theoretic foundation of circle geometry. Proc. Imp. Acad. Tokyo **16**, 515–517 (1940). [MF 3892]

The author gives a set of lattice-theoretic axioms for "circle geometry," in which the elements are points, pairs of points, circles, spheres, etc. He also shows how to define perspectivity and parallelism lattice-theoretically. No theorems are announced. Further extensions are sketched.

G. Birkhoff (Cambridge, Mass.).

Ziegenbein, P. Konfigurationen in der Kreisgeometrie. J. Reine Angew. Math. 183, 9–24 (1940). [MF 4225]

Using nothing more elaborate than the theorem that two intersecting circles meet twice at the same angle, the author establishes the following generalization of Miquel's theorem. Let (1), (2), $\cdots$, (n) be n circles passing through one point (O). Let (12) be the remaining point of intersection of (1) and (2), and let (123) be the circle through (12), (13), (23). Then the four circles (123), (124), (134), (234) pass through one point (1234); the five points (1234), (1235), (1245), (1345), (2345) lie on one circle (12345); and so on. This process leads to a circle-point-configuration $(2^{n-1}, n)$, consisting of 2^{n-1} points and 2^{n-1} circles, with n of the circles through each point, and n of the points on each circle. Moreover, the n circles through one point make the same angles as those through any other point of the configuration.

If the first n circles are drawn with the same radius, all the 2^{n-1} circles have the same radius, and their centers lie on a further set of 2^{n-1} circles of the same radius, forming a second configuration of the same kind. By applying an arbitrary inversion to the whole figure, we derive two singly-inverse (einfach-gespiegelte) configurations such that there is a point whose inverses with respect to the circles of either configuration are the points of the other.

The first place where the author resorts to the use of straight lines is in proving the existence (for every positive integer n) of two doubly-inverse configurations K_1 and K_2 such that there are two points F_1, F_2 whose inverses with respect to the circles of K_2, K_1 are the points of K_1, K_2, respectively. (Fig. 16 shows all the sixteen circles for the case $n=4$.) These determine an endless chain of configurations $\cdots$, K_{-1}, K_0, K_1, K_2, $\cdots$, such that the inverses of F_1, F_2 with respect to the circles of K_{r+1}, K_r are the points of K_r, K_{r+1}, respectively. Finally, for $n\leqq 5$, any configuration $(2^{n-1}, n)$ belongs to a doubly-inverse pair, and so to such a chain. *H. S. M. Coxeter* (Toronto, Ont.).

Merz, K. Heptaeder aus verschiedenen Netzen. Comment. Math. Helv. 13, 49–53 (1940). [MF 3987]

This seems to be a rather unsatisfactory elaboration of the non-orientable heptahedron (which consists of alternate faces of the octahedron, connected by three squares lying in planes through the center). *H. S. M. Coxeter.*

Merz, K. Kreuzhaube aus verschiedenen Netzen. Vierteljschr. Naturforsch. Ges. Zürich 85, 51–57 (1940). [MF 4396]

Let N be the center of a horizontal square $ABCD$, vertically above another square $A'B'C'D'$, and let O be vertically above N. Then the triangles and quadrangles ABN, BCO, CDN, DAO, $AOCN$, $BODN$, $ABB'A'$, $BCC'B'$, $CDD'C'$, $DAA'D'$, $A'B'C'D'$ form a one-sided polyhedron which, after removal of the base $A'B'C'D'$, will serve as a "cross-cap" in the construction of one-sided polyhedra of higher genus. The author elaborates this in the same way that he previously elaborated another one-sided polyhedron [cf. the preceding review]. *H. S. M. Coxeter.*

Hadwiger, H. Über ausgezeichnete Vektorsterne und reguläre Polytope. Comment. Math. Helv. 13, 90–107 (1940). [MF 4439]

The author defines a vector-star as a system of n vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ from a fixed origin in real s-dimensional space (but not lying in any subspace). The group of all congruent transformations of the star into itself is called the group of the star. The star is said to be transitive if its group is transitive on the vectors. It is said to be symmetrical if each vector is invariant under a non-identical subgroup. A particular transitive (but not symmetrical) star is the set of n mutually orthogonal unit vectors in n dimensions, like the edges at one vertex of the unit cube. The orthogonal projection of this on an s-space is called a Pohlke normal star, and is shown to be characterized by either of the following properties: (i) The s non-vanishing characteristic roots of the matrix of scalar products $||(\mathbf{a}_i, \mathbf{a}_k)||$ are all 1. (ii) For every vector $\mathbf{x}$ in the s-space, $\sum_1^n(\mathbf{a}_\nu, \mathbf{x})\mathbf{a}_\nu=\mathbf{x}$.

A set of n concurrent lines in s dimensions is called a P-bundle if it has any one of the three following properties, which are shown to be equivalent. (a) The sum of the squared cosines of the angles between these lines and a variable line is constant (and equal to n/s). (b) Vectors of magnitude $(s/n)^{\frac{1}{2}}$ along the lines form a Pohlke normal star. (c) The sum of the n projections of any vector $\mathbf{x}$ on the lines is the vector $(n/s)\mathbf{x}$. It is proved that such a bundle exists for all $n\geqq s$, and that a particular P-bundle consists of the lines of any transitive and symmetrical star, such as the star joining the center of a regular polytope to its vertices. [This result is checked, rather unnecessarily, by considering each regular polytope separately.] In virtue of (c), a theorem of Brauer and Coxeter [Trans. Roy. Soc. Canada, Sect. III (3) 34, 29–34 (1940); these Rev. 2, 125] would have enabled the author to establish the more general result that the lines of any transitive star whose group is irreducible form a P-bundle, and so provide a Pohlke normal star as in (b). *H. S. M. Coxeter* (Toronto, Ont.).

Convexities, Extremal Problems

Pauc, Christian. Sur la convexité dans les espaces distanciés. Revue Sci. (Rev. Rose Illus.) 78, 233 (1940). [MF 2451]

Three theorems are formulated concerning Menger's generalization of the conception of convexity to metrical spaces [Menger, Math. Ann. 100 (1928)]. The first two consider sequences of convex bodies; the second one generalizes Blaschke's "Auswahlsatz" [Blaschke, Kreis und Kugel, p. 62]. The last theorem deals with the existence and qualities of geodetics in metrical spaces which are locally convex. It contains a generalization of a theorem of Tietze [Math. Z. 28, 697 (1928)]. No proofs are given. *P. Scherk.*

Carleman, T. Sur les courbes paraboliquement convexes. Vierteljschr. Naturforsch. Ges. Zürich 85 Beiblatt (Festschrift Rudolf Fueter), 61–63 (1940). [MF 4405]

Let y be a rectifiable Jordan curve, the coordinates of which are four times differentiable with respect to the arc length, and the curvature of which vanishes nowhere. Then y possesses exactly one osculating parabola (o.p.) in each point. We call y parabolically convex if there is for each point of y a neighborhood on y that lies wholly in the closed interior of the o.p. of the point. An o.p. has at least four-point contact. If no o.p. has more than four-point contact, we call y properly parabolically convex (p.p.c.). The author proves: A p.p.c. closed curve is met by an o.p. only in its (unique) osculating point. He announces: A p.p.c. closed curve has no more than four points in common with any parabola. Analogous results hold for systems of curves depending upon an even number of parameters. *P. Scherk.*

Jessen, Børge. Two theorems on convex point sets. Mat. Tidsskr. B. **1940**, 66–70 (1940). (Danish) [MF 3882]

A new proof of the known theorem that a set of points in the n-dimensional Euclidean space is closed and convex if and only if it contains for every point of the complementary set a unique nearest point. Furthermore the author proves that a bounded, closed, convex set contains for every point of the complementary set a unique furthest point if and only if it has inner points and contains the mid-points of all osculating spheres of its boundary. *W. Feller.*

Kametani, Shunji. Some remarks on Kakeya-Kunugi's theorem. Jap. J. Math. **17**, 27–30 (1940). [MF 3154]

The author extends Kakeya-Kunugi's theorem [see Proc. Imp. Acad. Tokyo **13**, 296–300 (1937)] which gives an upper bound for the length of a Jordan curve under certain conditions. The length $\Lambda(B)$ of a plane point set B is defined as the limit as $\epsilon\to 0$ of the lower bound $\Lambda^{(\epsilon)}(B)$ of the sums $\sum_1^\infty \delta(B_i)$, where $B_1, B_2, \cdots$ is an arbitrary partition of B into a sequence of sets that have diameters $\delta(B_1), \delta(B_2), \cdots$ less than ϵ. Thus if B is a linear set, $\Lambda(B)$ is its Lebesgue outer measure; if B is a rectifiable continuous curve, $\Lambda(B)$ is its length in the ordinary sense. The paper contains three lemmas, a theorem and a corollary. The theorem establishes sufficient conditions that the length of an arbitrary plane point set be finite. From this theorem the author obtains the following corollary, which contains Kakeya-Kunugi's theorem: Suppose that with each point z of a continuous curve C lying in a circle of radius $R>0$ we can associate a sector of vertex z, radius $r>0$ and angle $\omega(0<\omega<\pi)$ such that it has no point in common with C except z. Then the length of C is not greater than a finite number depending only on ω, r and R. *G. B. Price* (Lawrence, Kan.).

Hirano, Kôtarô. Simple proofs of Vogt's theorem. Tôhoku Math. J. **47**, 126–128 (1940). [MF 4014]

The author gives two proofs of the following theorem of Vogt [J. Reine Angew. Math. **144**, 239–248 (1914)]: If the arc $\widehat{AB}$ has a positive curvature decreasing from A to B monotonically, and if the arc $\widehat{AB}$ and the line AB form an oval, then the angle between $\widehat{AB}$ and the line AB is greater in A than in B. A simpler proof of a slightly more general theorem has been given by Sutezo Katsuura [Tôhoku Math. J. **47**, 94–95 (1940); these Rev. **2**, 16]. Choosing the coordinates conveniently, the curve $\widehat{AB}$ may be given in the form $x=x(s)$, $y=y(s)$ $(0\leqq s\leqq 1)$ with $y(s)>0$ for $0<s<1$, $y(0)=y(1)=0$ and $k'=dk/ds<0$ (s=arc length; k=curvature). We have to prove $x'(0)>x'(1)$. Now

$$x'(1)-x'(0)=\int_0^1 x''(s)ds=-\int_0^1 ky'(s)ds=\int_0^1 k'yds<0.$$

P. Scherk (New Haven, Conn.).

Beretta, L. e Maxia, A. Sui vertici di un'orbiforme e sulle cuspidi della sua curva media. Boll. Un. Mat. Ital. (2) **2**, 216–220 (1940). [MF 2975]

The authors prove that the number of vertices (that is, points of extreme curvature) of a closed, convex curve of constant width is of the form $4n+2$ with $n\geqq 1$; here the radius of curvature is assumed to be continuous. The theorem had been proved by Ch. Jordan and R. Fiedler [Archiv Math. Phys. **22**, 362–364 (1914)] under the assumption that the radius of curvature is continuously differentiable. *F. John* (Lexington, Ky.).

Kubota, Tadahiko. Nachtrag zu meiner vorigen Arbeit "Ein Satz über Eilinien." Tôhoku Math. J. **47**, 177–180 (1940). [MF 4018]

[The paper referred to in the title appeared in the same vol., pp. 96–98; cf. these Rev. **2**, 12.] Let the two ovals c and k have a continuous radius of curvature; in the relative differential metric with k as standard curve (Eichkurve), denote by ρ_A and ρ_B the minimal and the maximal relative radius of curvature of c, which may be reached at A and B, respectively. Any relative circle which intersects c in at least 3 (different or coinciding points) has a radius ρ with $\rho_A\leqq\rho\leqq\rho_B$. It follows from this that the minimal relative circle of curvature is contained in c, and that the maximal one contains c. *H. Busemann* (Chicago, Ill.).

Santaló, L. A. A theorem on sets of parallelepipeds with parallel edges. Publ. Inst. Mat. Univ. Nac. Litoral **2**, 49–60 (1940). (Spanish) [MF 3009]

Given a set of (n-dimensional) parallelepipeds in Euclidean n-space; their edges shall be parallel to n fixed directions. For any subset of $2^{n-1}(n+1)$ or $2^{n-1}(2n-1)$ parallelepipeds there will exist an $(n-1)$-space or a straight line, respectively, which meets each parallelepiped of the subset. Then there exists an $(n-1)$-space or a straight line, respectively, which meets all the parallelepipeds of the set. The proof is based upon an idea of Radon [Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten, Math. Ann. **83**, 113–115 (1921)]. [The transition from finite to infinite sets of parallelepipeds is done only for $n=2$ and $n=3$; but analogous arguments obviously hold in the general case.] *P. Scherk* (New Haven, Conn.).

Dinghas, Alexander. Verallgemeinerung eines Blaschkeschen Satzes über konvexe Körper konstanter Breite. Rev. Math. Union Interbalkan. **3**, 17–20 (1940). [MF 2673]

Let K be a convex solid in n-space. The volume of the parallel solid of distance h is given by an expression $\sum_{\nu=0}^n \binom{n}{\nu} W_\nu h^\nu$ [see Bonnesen-Fenchel, Theorie der konvexen Körper, Ergebnisse der Mathematik, Bd. 3, Heft 1, 1934, p. 49]. The author proves that, in the case of a solid K of constant breadth b, the coefficients W_ν satisfy the recursion formula

$$2W_{n-q}=\sum_{\nu=0}^{q-1}(-1)^\nu\binom{q}{\nu}W_{n-\nu}b^{q-\nu}$$

for every odd $q\leqq n$. This is a generalization of a result by Blaschke for $n=3$. *F. John* (Lexington, Ky.).

Dinghas, Alexander. Zur Theorie der konvexen Körper im n-dimensionalen Raum. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 4, 30 pp. (1939). [MF 1085]

If K is a convex body in n dimensions, and $W_0, W_1, \cdots, W_n$ its mixed volumes with the unit n-sphere, then the Minkowski inequality is that $W_nW_{n-1}\leqq W_{n-1}^2$. In the case $n=2$, this reduces to the isoperimetric inequality and in the case $n=3$ to the inequality between the surface area and the integral of the mean curvature. The inequality becomes the equality in the case K is an n-sphere. The author in the present paper engages in sharpening the inequality if K is not an n-sphere. The results are obtained for $n=2, 3$ and

then for a general n. He first proves that, if K is a body such that the projection on some $(n-1)$-plane is an $(n-1)$-sphere, as for example a body of revolution, then

$$\text{(a)}\quad W_n W_{n-2} = W_{n-1}^2 - (W_{n-1} - W_n d)^2 - [1/n(n-1)]\cdot Q(p-d).$$

Here p is the Stützfunktion of K, $d\omega_n$ is the element of area on the unit n-sphere, $2d$ is the greatest width of K normal to the axis of rotation, ∇_1 is the first Beltrami operator on the unit n-sphere, and

$$Q(p-d) = \int [\nabla_1(p-d) - (n-1)(p-d)^2] d\omega_n,$$

which is not negative. The proof is made with the use of integral identities. If K is not such a body, a process of symmetrizing is performed by which K is transformed into a body of revolution K' with the same W_{n-1} and a W_{n-2} which is not smaller. Since (a) holds for K' and W_n is the volume of the unit n-sphere, (a) holds also for K, except that $=$ is replaced by $\leqq$. The p and d in (a) will now refer to K'.

A number of special inequalities can be obtained from (a). If the last term in (a) be dropped, an inequality results which in the case $n=2$ reduces to a sharper form of the isoperimetric inequality $A \leqq L\cdot d - \pi d^2$ previously obtained by E. Schmidt. *J. W. Green* (Rochester, N. Y.).

Dinghas, Alexander. Elementarer Beweis einer Ungleichung für konvexe Körper. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 9, 20 pp. (1939). [MF 1086]

If O, V and M are the surface area, volume and integral mean curvature of a convex body K in three dimensions, three special cases of Minkowski's inequalities give $4\pi O \leqq M^2$, $3MV \leqq O^2$ and $48\pi^2 V \leqq M^3$. In the case K is a body of revolution, Bonnesen has sharpened these to (a) $4\pi O \leqq M^2 - (M-4\pi d)^2$, (b) $MV \leqq O - (O-Md)^2$ and (c) $48\pi^2 V \leqq M^3 - (M-4\pi d)^2\cdot(M+8\pi d)$, where $2d$ is the distance between two parallel supporting planes of K parallel to the axis of revolution. In the present paper the author gives a more elementary proof of (a), (b) and (c). In addition, he gives a proof of (c) in the case K is an arbitrary convex body, except that d is replaced by $\bar{d}$, where $2\bar{d}$ is the distance between some two (but not any two) parallel supporting planes for K; furthermore a direction to which the planes must be parallel may be assigned in advance. The author had already given a similar result for (a) in the case of a general convex body. *J. W. Green* (Rochester, N. Y.).

Dinghas, Alexander. Beweis einer Ungleichung für konvexe Körper. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 11, 13 pp. (1939). [MF 1491]

If K_1 and K_2 are two convex bodies in three dimensions, then the surface area of the body $(1-t)K_1 + tK_2$ is equal to $(1-t)^2 O_{11} + 2(1-t)tO_{12} + t^2 O_{22}$, where the O_{ij} are the mixed surfaces of K_1 and K_2. In particular O_{11} is the area of K_1 and O_{22} the area of K_2. The author gives an elementary proof of the Minkowski inequality $O_{11}O_{22} \leqq O_{12}^2$. He makes use of integral inequalities, and avoids the use of spherical harmonics. *J. W. Green* (Rochester, N. Y.).

Dinghas, Alexander. Konvexe Rotationskörper im n-dimensionalen Raum. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 17, 26 pp. (1939). [MF 3755]

If K is a convex body of revolution in three dimensions, Bonnesen has obtained and the author has given simple proofs [see the second review above] for a pair of equations (a) $3V = 2\cdot O\cdot d - Md^2 - P$ and (b) $O = 2Md - 4\pi d^2 - Q$, where V, O and M are the volume, area and integral mean curvature, respectively, of K, d is the greatest width of K normal to the axis of rotation and P and Q are non-negative quantities depending on K. If V, O and M are given their values in terms of the mixed volumes of K with the unit sphere, namely W_0, $3W_1$ and $3W_2$, respectively, the equations (a) and (b) become $W_1 - 2W_1 d + W_2 d^2 = -\frac{1}{3}P \leqq 0$, and $W_1 - 2W_2 d + W_3 d^2 = -\frac{1}{3}Q \leqq 0$. Generalizing from 3 to n dimensions, the author conjectures and proves the existence of a set of $n-1$ equations for a convex n dimensional body of rotation: $W_{i-1} - 2W_i d + W_{i+1} d^2 = -P_i$, $i = 1, 2, \cdots, n-1$, $P \geqq 0$. These equations give as a special case the known inequalities of Fenchel, $W_0/W_1 \leqq W_1/W_2 \leqq \cdots \leqq W_{n-1}/W_n$, and also a number of other known inequalities.
J. W. Green (Rochester, N. Y.).

Haupt, Otto. Über eine Kennzeichnung der Kugel. Jber. Deutsch. Math. Verein. **50**, 113–120 (1940).

The author calls a point set "kreiskonvex" if there is for any three points of the set an arc of a circle or a segment of a straight line which belongs to the set and contains them. He determines the "kreiskonvex" sets of the n-sphere and proves especially that every bounded and "kreiskonvex" point set of Euclidean n-space with interior points has an $(n-1)$-sphere as its boundary. *P. Scherk.*

Schmidt, Erhard. Die isoperimetrischen Ungleichungen auf der gewöhnlichen Kugel und für Rotationskörper im n-dimensionalen sphärischen Raum. Math. Z. **46**, 743–794 (1940). [MF 3379]

This paper is a continuation of the author's papers in Math. Z. **44**, 689–788 (1939) and Math. Z. **46**, 204–230 (1940) [these Rev. **2**, 12], which deal with the isoperimetric inequality in Euclidean and hyperbolic spaces of constant curvature. Let C be a simple, closed curve on the unit sphere S of length $L < 2\pi$, and let J be the smaller area bounded by C on S. Let $2R$ be a "breadth" of C, that is, let C be inscribed to a parallel strip bounded by the two circles of (spherical) distance R from a great circle. Let $\bar{J}(r)$ and $\bar{L}(r)$ denote the functions giving, respectively, the area and circumference of a circle of radius r on S. The author derives the inequality

$$\text{(A)}\qquad J \leqq \frac{\bar{J}'(R)}{\bar{L}'(R)} L - \left\{ \frac{\bar{J}'(R)}{\bar{L}'(R)} \bar{L}(R) - \bar{J}(R) \right\};$$

if R_1 is the radius of the circle of circumference L, then for fixed $R \leqq R_1$ and L the equality sign in (A) holds only for one curve. The ordinary isoperimetric inequality

$$\text{(B)}\qquad J \leqq \bar{J}(R_1),$$

where equality occurs only for circles, is a consequence. The same inequalities are derived for hypersurfaces of revolution in n-dimensional elliptic space of constant curvature, J and L denoting the corresponding volumes and "surface areas." Also the inequalities (A) and (B) are the same as those derived for Euclidean and hyperbolic geometry in the papers mentioned above. *F. John* (Lexington, Ky.).

John, Fritz. An inequality for convex bodies. Univ. Kentucky Research Club Bull. **1940**, no. 6, 26 (1940). [MF 3198]

Let c be a plane convex region, D its diameter and d its width (D and d are respectively the greatest and smallest

distance of two parallel lines of support). The region c is called "round" if $D/d \leqq \sqrt{2}$. Theorem: If P is an interior point of the 3-dimensional convex body C, then there exists a plane section c of C through P, which is round. Moreover there is a plane π such that the normal projection of C on π is round. The proof rests on the author's previous result [Duke Math. J. 2, 447–452 (1936)] that a sufficient condition for the roundness of c is that the central ellipse of inertia of c be a circle and uses elegantly the topological impossibility of a continuous field of tangents on a sphere.

I. J. Schoenberg (Waterville, Me.).

Szekeres, Gy. On an extremum problem in the plane. Amer. J. Math. 63, 208–210 (1941). [MF 3643]

In a recent paper [Amer. J. Math. 61, 912–922 (1939); these Rev. 1, 35] the reviewer proposed the following problem: What is the minimum of the maximum of the $3\binom{n}{3}$ angles formed by a set M of n points of a plane as M describes all planar subsets of n points? It was stated in the paper referred to that for $n=3, 4, 5, 6$ the answer to the query is furnished by the function $(1-2/n)\pi$, but that for $n=7$ this formula fails to give the desired minimum maximum angle. In the paper under review the author establishes the following two interesting results: (1) For $n \geqq 2$ there exist planar sets of 2^n points such that all the angles formed are less than $(1-1/n)\pi+\epsilon$, with ϵ arbitrarily small. (2) Among the angles formed by 2^n+1 points of the plane there is at least one which exceeds

$$\pi\left(1-\frac{1}{n}+\frac{1}{n(2^n+1)^2}\right).$$

For the analogous problem in three dimensional space, the author states that among 2^n+1 points there are always three which form an angle exceeding $(1-c_1/\sqrt{n})\pi$, while on the other hand there exist 2^n points such that the maximum angle is less than $(1-c_2/\sqrt{n})\pi$, where c_1 and c_2 are constants.

L. M. Blumenthal (Columbia, Mo.).

MECHANICS

Dynamics, Celestial Dynamics

Siegel, Carl Ludwig. Der Dreierstoss. Ann. of Math. (2) 42, 127–168 (1941). [MF 3678]

The author discusses the motion of three particles of masses m_1, m_2, m_3 under the Newtonian law of gravitation. If at time t they are at points A_1, A_2, A_3, if the distances A_2A_3, A_3A_1, A_1A_2 are denoted by r_1, r_2, r_3, respectively, and if the rectangular coordinates of the points are x_k, y_k, z_k $(k=1, 2, 3)$, then these variables satisfy well-known differential equations. If these coordinates and their derivatives with respect to t have assigned finite values at $t=t_0$ such that no two of the points coincide, they are regular analytic functions of t in the vicinity of $t=t_0$. Consider the analytic continuation of these functions along the real axis of t. The author discusses the case which arises if there is a time $t=t_1$ at which these functions have a singular point. Without loss of generality he assumes that $t_1=0$, and takes $t>0$. He discusses the character of the functions as $t\to 0$. Two possibilities occur: (1) one of the r's approaches zero while the other two r's do not; (2) all three of the r's approach zero. This latter case is called "Dreierstoss" by the author, and he analyzes this case. Sundman had treated the same problem less fully in a famous paper [Acta Soc. Sci. Fennicae 34 (1907)]. Sokoloff obtained most of the results given in this paper by Siegel [Acad. Sci. Ukraine, Memoires 9 (1928)].

Writing $r_k=\bar{r}_k t^{2/3}$, we discover that as $t\to 0$ the $\bar{r}$'s approach positive limits and that either (1) these limits are equal, or (2) one of them is the sum of the other two. These are referred to as the equilateral triangle case and the straight line case, respectively. It is found that as $t\to 0$ the direction angles of the lines A_1A_2, A_2A_3 and A_3A_1 approach limiting values. By translation and rotation of coordinate axes with constant velocities, the motion is reduced to that of motion in a plane, so that we may take $z_k=0$. It is found that in the equilateral triangle case there are two physically significant independent parameters, and in the straight line case two such parameters. The analytic character of the x's and y's as functions of t depends upon the relative magnitudes of the m's. The author gives formulas defining three numbers a_1, a_2, b_1 as functions of the m's in terms of which results are easily stated. In the equilateral triangle case, if we write

$$x_k=t^{2/3}x_k^*(u_1, u_2, u_3), \quad y_k=t^{2/3}y_k^*(u_1, u_2, u_3),$$

then (1) if $2/3a_2$ or a_1/a_2 is an integer the functions x_k^* and y_k^* are power series in $u_1=\alpha_1 t^{2/3}$, $u_2=\alpha_2 t^{a_1}$, $u_3=\alpha_3 t^{a_2}$ whose coefficients depend only on the m_k's; or (2) under other conditions replace this u_1 by

$$u_1=(\alpha_1+c_1\alpha_3^g \log t)t^{2/3}$$

or u_2 by

$$u_2=(\alpha_2+c_2\alpha_3^h \log t)t^{a_1}.$$

Similar results are obtained for the straight line case.

E. J. Moulton (Evanston, Ill.).

Subbotin, M. F. Sur quelques propriétés du mouvement dans le problème de n corps. C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 440–442 (1940). [MF 3222]

Starting from a formula discovered by Lagrange in discussing the motion of three bodies, a formula extended by Jacobi to the motion of n bodies, the author transforms the equation by a substitution and arrives at formulas which are quite similar to formulas for the problem of two bodies. He obtains a generalization of Kepler's equation and also Lagrange's particular solution of the problem of n bodies.

E. J. Moulton (Evanston, Ill.).

Herrick, Samuel, Jr. The Laplacian and Gaussian orbit methods. Univ. California Publ. Contrib. Los Angeles Astr. Dept. 1, 1–56 (1940). [MF 3937]

This paper deals with the determination of an unconditioned orbit from three observed positions separated by intervals of time sufficiently short for series expressions to hold. The author points out that, while the coordinates of the observed body are defined at every instant by means of the equations of motion, the coordinates of the observer are defined only for the three instants of observation. He then defines the coordinates of the observer for other instants in such a way that the remaining quantities which enter the fundamental equations of the Laplacian method can be expressed exactly in terms of the six observed quantities. This procedure eliminates, for each successive approximation, the recomputation of the determinant of the

equations and of all the other determinants except one. The series developments which result have three forms depending upon the variables chosen. One form is precisely that used in the Gaussian method. Several other interesting procedures are developed, and examples are worked out. There is derived a basis for the comparison of the various methods of determining an unconditioned orbit.

L. E. Cunningham (New Haven, Conn.).

Chazy, Jean. Sur une loi corrective de la loi de Newton. C. R. Acad. Sci. Paris **210**, 713–716 (1940). [MF 3917]

The author derives the approximate variations of the orbital elements of a planet revolving around a rotating homogeneous spherical central body on the assumption of a modified law of gravitational attraction

$$F = -(fmm'/r^2)(1+\epsilon dr/dt).$$

This is an extension of an earlier note by the same author [C. R. Acad. Sci. Paris **209**, 133–136 (1939); these Rev. **1**, 20].

D. Brouwer (New Haven, Conn.).

Godart, O. On space closure of periodic orbits in the field of a magnetic dipole. J. Math. Phys. Mass. Inst. Tech. **20**, 207–217 (1941). [MF 4127]

Periodic orbits in the meridian plane have been studied by Störmer [Z. Astrophys. **1**, 237–274 (1930)] and by Lemaître [Ann. Soc. Sci. Bruxelles (A) **54**, 194–207 (1934)]. In general such orbits are not closed in space, but fill a surface of revolution having as meridian section the periodic orbit in the meridian plane. In this paper it is shown that periodic orbits are closed in space if the ratio between the period in the meridian plane, and the period of rotation of this plane is commensurable. This condition is satisfied for a certain denumerable set of values of Störmer's parameter γ_1. Tables are given for the values of the Fourier coefficients of both stable and unstable principal periodic orbits, from which the values of γ_1 and the Fourier coefficients for periodic orbits closed in space can be calculated. An example is worked out for such an orbit having one turn and three oscillations, for which $\gamma_1 = 0.789779$. Such orbits play an important role in the theory of primary cosmic rays.

M. S. Vallarta (Cambridge, Mass.).

Störmer, Carl. Sur la recherche qualitative et quantitative d'un système d'équations différentielles jouant un rôle important dans la physique cosmique. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 310–315. [MF 3523]

A review of Störmer's work on the motion of a charged particle in the field of a magnetic dipole. No new results are given and no mention is made of the advances made by other investigators since 1933. *M. S. Vallarta.*

Bhatnagar, P. L. On the origin of solar system. Indian J. Phys. **14**, 253–281 (1940). [MF 3939]

In this investigation the tidal theories of the origin of the solar system have been reexamined from the point of view of the forces necessary to produce the required planetary filaments. The method consists in studying the variation of the tidal field on the surface of a star (the "primary") by another star (the "secondary") and determining how close the approach of the two stars should be in order that the primary may become unstable. This distance of minimum approach is seen to depend on the ratio of the masses only. Thus, if the distance of closest approach is assumed to be 20 times the radius of the primary, then for tidal disruption the mass of the secondary will have to be 485 times the mass of the primary. When these results are further modified to take into account the relative motion of the two stars, the circumstances become even less favorable for the formation of the planetary filaments. Lyttleton's version of the tidal theory in which the sun is assumed to be originally the component of a double-star system is also examined. It again appears that under the most favorable circumstances for the formation of the planetary filaments the sun would come so near its companion and the visiting star that a collision between them can hardly be avoided. On the strength of these arguments the author concludes that "no existing tidal theory can satisfactorily explain the origin of the solar system." *S. Chandrasekhar.*

Relativity

Fahmy, M. The idea of minimum proper time, and some consequences of it. Philos. Mag. (7) **30**, 331–339 (1940). [MF 3184]

Kaluza's five-dimensional theory of relativity has been modified by Fisher, Flint and others to give a representation of the path of an electron as a minimal geodesic in a five-dimensional hyperspace; comparison with quantum theory leads to the suggestion that the fifth coordinate is quantized in multiples of h/m_0c, where m_0 is the electron mass. Fahmy investigates the implications of these hypotheses for time and space intervals associated with the motion of any particle in an electrostatic field. He concludes that "the dynamical description of the atomic orbits is impossible if they are regarded as occupied by particles more massive than the electron," that there is a least distance to which a charged particle can approach an unlike charge, but that this restriction is inoperative in case the two charges have the same sign. *H. P. Robertson* (Princeton, N. J.).

Haskey, H. W. Einstein's distant parallelism and Dirac's equation. Philos. Mag. (7) **30**, 478–486 (1940). [MF 3710]

It is known that by using an orthogonal enuple $h_a{}^\mu$ such that $\delta^{ab}h_a{}^\mu h_b{}^\nu = \gamma^{\mu\nu}$ one may obtain from constant matrices satisfying $\frac{1}{2}(E_aE_b+E_bE_a) = \delta_{ab}$ variable matrices $\gamma^\mu = h_a{}^\mu E_b\delta^{ab}$ which satisfy $\frac{1}{2}(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu) = \gamma^{\mu\nu}\cdot 1$. The author attempts to prove from this that "the Dirac's equation for a Gallilean five world is precisely the same form as Flint's in a Riemannian five-world over any given small region." The proof seems inadequate. *A. H. Taub* (Princeton, N. J.).

Eigenson, M. S. Cosmological relativity and relativistic cosmology. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 751–753 (1940). [MF 3575]

Sibata, Takasi. Wave geometry unifying Einstein's law of gravitation and Born's theory of electrodynamics. II. J. Sci. Hirosima Univ. Ser. A. **10**, 157–171 (1940). [MF 3383]

Some misstatements in a previous paper of the author's [J. Sci. Hirosima Univ. Ser. A. **8**, 51–79 (1938)] are corrected. The motive of the two papers is to write equation (4) of the preceding papers so that the conditions of integra-

bility of these equations imply Einstein's field equations for free space and Born's electromagnetic equations.
A. H. Taub (Princeton, N. J.).

Schrödinger, Erwin. Maxwell's and Dirac's equations in the expanding universe. Proc. Roy. Irish Acad. Sect. A. 46, 25–47 (1940). [MF 3053]

The author solves [§ 2] Maxwell's equations for free space in a cosmological space-time, whose minimal invariant varieties are spaces of constant Riemannian curvature $1/R^2(t)$. By dealing with the first order equations for the field strengths, instead of the second order equations for the vector potential, he is able to obtain a general solution in terms of the spatial coordinates and the "cosmic time" $\tau=\int cdt/R(t)$. The solutions contain a factor $e^{i\nu t}$, and boundary conditions lead to the condition that the "frequency" ν must be an integer of absolute value not less than 2. Special properties of the solutions are obtained in Appendices I–V. The Dirac equations are discussed [§ 3] for the same space-time, in a slightly different coordinate system. The spatial dependence is taken from a previous [Pont. Acad. Sci. Comment. **2**, 321 (1938)] paper by the same author, which dealt with the case in which R is independent of t; the time dependence of the solution for the special case $R(t)=a+bt$ is given in Appendix VI. [These solutions of the Dirac equations were obtained by A. H. Taub, Phys. Rev. **51**, 512–525 (1937).] *H. P. Robertson* (Princeton, N. J.).

Haantjes, J. Die Gleichberechtigung gleichförmig beschleunigter Beobachter für die elektromagnetischen Erscheinungen. Nederl. Akad. Wetensch., Proc. 43, 1288–1299 (1940). [MF 3942]

The author examines the transformation properties of the retarded potential under the four-dimensional conformal transformations, which were shown by H. Bateman [Proc. London Math. Soc. (2) **8**, 223 (1910)] and E. Cunningham [Proc. London Math. Soc. (2) **8**, 77 (1910)] to leave Maxwell's equations invariant. It is then shown that the world line of a particle moving with a constant acceleration along a straight line, which is such that it is at rest at the origin of coordinates of an observer B at time $t=0$, can be transformed by a particular four-dimensional conformal transformation into the world line of a particle at rest with respect to B. This transformation has been given by L. Page [Phys. Rev. (2) **49**, 254–268 (1936)]. The relation between the conformal group and a general kinematic theory has been discussed by H. P. Robertson [Phys. Rev. (2) **49**, 755–760 (1936)] in a paper on Page's work. *A. H. Taub.*

Synge, J. L. A modified electromagnetic energy-tensor. Trans. Roy. Soc. Canada. Sect. III. (3) 34, 1–27 (1940). [MF 3349]

An electromagnetic field is considered to be created by point charges characterized by a constant electric charge e and an energy momentum vector M_r which may be related to a velocity vector λ_r by the equation $M_r=m\lambda_r$, where m is an invariant. The field due to a single particle is assumed to be derived in the usual manner by the retarded potential. That due to a collection of particles is the sum of the individual fields. The energy tensor of the field due to N charged particles is assumed to be

$$T_{rs}=c^{-2}\sum_{p\neq q}(\overset{(p)}{F}_{mr}\overset{(q)}{F}_{ms}-\tfrac{1}{4}\delta_{rs}\overset{(p)}{F}_{mn}\overset{(q)}{F}_{mn}),$$

instead of the usual expression

$$O_{rs}=c^{-2}(F_{mr}F_{ms}-\tfrac{1}{4}\delta_{rs}F_{mn}F_{mn}),$$

where

$$F_{rs}=\sum_{1}^{N}\overset{(p)}{F}_{rs}.$$

Thus the "self-energy" terms are omitted in the energy. Energy and momentum are assumed to be conserved in the sense that the total flux of 4-momentum across any closed 3-space is zero. The consequences of the above assumptions are developed in detail. The equations of motion of a particle are derived and contain no "radiation term." The total energy and momentum are computed by integrating over a null-cone instead of over a flat section of space-time, and finite expressions are obtained. However, the flux of energy and momentum at infinity is not zero. That is, the total energy is not constant. The energy for the case of two particles is obtained. *A. H. Taub* (Princeton, N. J.).

Hydrodynamics, Aerodynamics, Acoustics

Jaeger, Charles et Abecasis-Manzanares, Alberto. Le théorème de la simultanéité du minimum de l'énergie totale et du débit maximum dans le cas d'un écoulement plan permanent à filets courbes. C. R. Acad. Sci. Paris 210, 729–731 (1940). [MF 3919]

If H is the average total head across the depth h of a stream with the flow Q per unit time, the relation $F(H, h, Q)=0$ exists for any given locality of the stream. Bélanger stated that, for flow over a spillway with broad crest, the top and bottom surfaces of the stream are parallel and $\partial Q/\partial h=0$, that is, the flow is a maximum. Boess [Berechnung der Wasserspiegellage, Berlin, 1919, pp. 20 and 52] has shown that, for the so-called "critical section" with parallel flow, $\partial H/\partial h=0$, that is, the total head is a minimum. The present authors use the relation $F(H, h, Q)=0$ to demonstrate the reciprocity of these two statements.
H. S. Tsien (Pasadena, Calif.).

Bjerknes, V. On the motion of fluid bodies through fluids. I. Arch. Math. Naturvid. 43, 51–66 (1940). [MF 3239]

The author calls fluid bodies limited parts of a fluid system which are, or have been, under the influence of external forces. Their dynamics, which may prove to be of considerable significance for meteorology and oceanography as well as for the physics of the sun, are developed by the author under restriction to nonviscous incompressible fluids. The author shows that the integration of the equations can be substantially simplified by considering the velocity-vector v as the geometric sum of two vectors $\bar{v}$ and v^*, the first of which is defined (in Gibbs' vectorial notation) by the equation

$$\frac{\partial\bar{v}}{\partial t}=-\nabla(p+\tfrac{1}{2}\bar{v}^2)$$

(p=pressure belonging to the total velocity field v) and the initial condition $\bar{v}=0$ for $t=0$. The author shows that $\bar{v}$ is irrotational and that the pressure p can be eliminated from the dynamic fundamental equation, which finally can be brought into the following form:

$$\frac{dv^*}{dt}=f'-v^*\nabla\bar{v},$$

where f' is the external (active) mass-force, while $-v^*\nabla\bar{v}$ is the "hydrodynamic force of inertia." This equation combined with the equation of continuity is basic for the solutions offered by the author. He is particularly interested in the resulting force acting upon the fluid bodies; in the evaluation of these he is aided by an analogy with magnetism in which the place of v^* is taken by the "impressed magnetization," while that of $\bar{v}$ by the intensity of the magnetic field. *P. Nemenyi* (Iowa City, Iowa).

Pellew, Anne and Southwell, R. V. On maintained convective motion in a fluid heated from below. Proc. Roy. Soc. London. Ser. A. **176**, 312–343 (1940). [MF 3647]

The stability of a portion of an incompressible liquid contained between two horizontal planes and a cylindrical vertical surface is investigated. The cylinder may be either a solid boundary or a surface of symmetry separating adjacent liquid cells; in either case it is assumed that no heat is transmitted through it. Its cross-sectional shape is not assumed. The top and bottom planes may be either free surfaces or solid boundaries; it is assumed that they are externally cooled and heated, respectively, so that their temperatures are kept constant. Deviations from the equilibrium state of the liquid are investigated. These are assumed to be small, so that second-order terms in the velocity components and in the deviations of temperature, pressure and density from the equilibrium state are negligible. It is also assumed that the total difference in density between the top and bottom in the initial state is small. First letting the vertical component w of velocity be proportional to an exponential function of the time, it is shown that the exponent must have a negative real part unless it is purely real; that is, oscillatory motions cannot exist unless they decay. Hence the limiting conditions of stability are obtained when the exponent is zero and all time variations vanish.

The effect of the cell planform appears in a parameter a^2 which may be interpreted as being proportional to the natural frequency of a uniform elastic membrane having the shape of the cell cross section and vibrating under certain boundary conditions. For the various top and bottom conditions under consideration the values of the factor λ that relates a and the "characteristic number" $\lambda^3a^4 = gh^3\Delta\rho/\rho k\nu$ are determined (where h is the depth of the fluid, $\Delta\rho/\rho$ the fractional excess of density at the top surface over that at the bottom in the equilibrium condition, k the diffusivity, ν the kinematic viscosity and g the acceleration due to gravity). For the case (a) of two free horizontal surfaces, a simple expression for λ involving a is obtained; for (b) two solid boundaries, transcendental equations are obtained; these are solved numerically for several values of a. The values for the case (c) of one free and one solid boundary can be deduced from those for case (b). For any fixed cell planform (solid walls), the values of a can be found. [This problem is not treated in general in the paper.] The smallest of the corresponding values of λ^3a^4 provides the stability criterion. If the fluid is indefinite in extent, the cells may be triangular, rectangular or hexagonal. A particular size (corresponding to the value of a for least λ^3a^4) is associated with each shape, but the present theory does not indicate which shape is more likely to occur. Simple expressions for a are obtained in these cases. The hexagon, which is especially interesting in view of Bénard's observations, is treated by an exact method instead of Rayleigh's approximate one.

Finally an approximate treatment analogous to the use of "Rayleigh's principle" in vibrations is proposed for the calculation of λ in the various cases. It is shown that any reasonably close approximation to the form of w will lead by this method to a close estimate of λ. The method is illustrated by numerical calculations for case (b).
W. R. Sears (Inglewood, Calif.).

Burgers, J. M. Some considerations on the development of boundary layers in the case of flows having a rotational component. Nederl. Akad. Wetensch., Proc. **44**, 13–25 (1941). [MF 4042]

The hydrodynamical equations for a rotating system of axes and a frictional force per unit mass are specialized for flow in a boundary layer of thickness δ, the first equation being

$$\rho^{-1}\partial p/\partial y = \omega^2 r\partial r/\partial y - 2(\omega_z u - \omega_x w),$$

which shows that within the boundary layer p can vary at most by an amount of order δ. When this variation is neglected the value of p to be used within the boundary layer is determined by the flow outside the boundary layer which is not influenced by frictional forces. The condition of no vorticity relative to fixed axes for the external region is replaced by one specifying the curl of the velocity and a corresponding expression is found for the pressure. This is then used to transform the equations for the interior of the boundary layer. A study is then made of the flow in the boundary layer along a surface of revolution for the case in which the motion is wholly symmetrical with respect to the axis of motion. Special attention is paid to the case of a tube of axial symmetry with a body of revolution forming a partial obstruction. Centrifugal forces are found to play an important part in the boundary layers of rotating pumps and ventilators. *H. Bateman* (Pasadena, Calif.).

Cagniard, M. L. Sur la propagation du mouvement dans les milieux visqueux. Ann. Physique **13**, 239–265 (1940). [MF 3975]

A natural mixed problem for the wave equation with a viscosity term is (1) $a^2u_{xxt}+V^2u_{xx}-u_{tt}=0$, u, $u_t=0$ for $t=0$, $u=F(t)$, $F(0)=F'(0)=0$ for $x=0$. It is known that when $a\neq 0$ there is no wave front, that is, discontinuities in derivatives of u are not propagated, and the effect of $F(t)$ is not localized. The author's interest is in giving a physically convenient definition for the transition layer, taking the place here of a wave front, and he treats also asymptotic properties of u for a, $V\to 0$ and x, $t\to\infty$. The mathematical methods are conventional and the treatment is partly heuristic. The Carson formulation of the Heaviside calculus is used and attention is focused on the indicial admittance $A(v,\xi)$, where v, ξ are parameters taking the place of x, t and involve a^2 and V^2. The transition layer is defined as the range over which $\partial A/\partial v$ differs "appreciably" from 0. The saddle point method gives the asymptotic form of $A(v,\xi)$ as a constant plus an erf function and indicates that asymptotically there is propagation of a pseudo-wave with velocity V. Energy transfers are not discussed.
D. G. Bourgin (Princeton, N. J.).

Lamla, Ernst. Die symmetrische Potentialströmung eines kompressibeln Gases um einen Kreiszylinder im Kanal im unterkritischen Gebiet. Luftfahrtforschung **17**, 329–331 (1940). [MF 3959]

The symmetrical two-dimensional flow of a compressible fluid about a circular cylinder whose center lies on the

centerline of a channel with parallel walls is investigated by the successive-approximation method of Janzen and Rayleigh. The calculation is carried to the "second approximation"; that is, the terms in α^2 are found, where $\alpha = U/a_0$, U is the undisturbed velocity and a_0 is the velocity of sound at rest. The flow in the "first approximation" (incompressible fluid) is found by the method of images, which causes a certain small distortion of the cylinder. [Reviewer's note: An error is made in a numerical calculation at the end of the paper. It has been corrected by the author in a supplement to the paper [Luftfahrtforschung 18, 37 (1941)].]

W. R. Sears (Inglewood, Calif.).

Ringleb, Friedrich. Über die Differentialgleichungen einer adiabatischen Gasströmung und den Strömungsstoss. Deutsche Math. 5, 377–384 (1941). [MF 4248]

Consider a stationary adiabatic two-dimensional nonviscous flow. Let w, θ, ρ, c be the two-dimensional vector velocity, the polar angle, the density and the sound velocity, respectively. The restriction to an irrotational source-free flow implies $w = \text{grad}\,\Phi$ and $\rho w/\rho_0 = \text{curl}\,\psi$. The analysis, though elementary and amounting to changing independent variables in differentials and partial derivatives and simple eliminations, leads to striking results. $\psi(|w|, \theta)$ satisfies a second order partial differential equation, which is elliptic or hyperbolic depending on the relative magnitudes of c and $|w|$. The central formula of the paper is the expression for the acceleration $d|w|/dt$ along a flow line ($d\psi = 0$), namely,

$$d|w|/dt \sim \psi_\theta/(c^{-2} - |w|^{-2})\psi_\theta^2 - \psi^2_{|w|}.$$

Critical values are those for which the acceleration becomes infinite. For $c < |w|$, $\psi_\theta \neq 0$, there are real critical curves and it is shown that the flow lines must then have cusps. For $c \geqq |w|$ there may be critical points, etc. [Evidently such facts are implicit in the nature of the characteristics of the partial equation mentioned, but this observation is not made.]

D. G. Bourgin (Princeton, N. J.).

Görtler, H. Über eine dreidimensionale Instabilität laminarer Grenzschichten an konkaven Wänden. Nachr. Ges. Wiss. Göttingen. Fachgruppe I. (N.F.) 2, 1–26 (1940). [MF 3970]

In a previous paper by the author [Z. Angew. Math. Mech. 20, 138–147 (1940); these Rev. 2, 170] the boundary layer flow over a concave wall is found to be more stable than that over a convex wall under infinitesimal two-dimensional disturbances in a plane perpendicular to the axis of the cylindrical wall. However, in a paper by G. I. Taylor [Philos. Trans. Roy. Soc. London, Ser. A. 223, 289–343 (1923)] treating the problem of stability of flow between two rotating cylinders, it is shown that the breakdown of laminar flow is caused not by the two-dimensional disturbance, but by a three-dimensional disturbance consisting of alternating vortex rings around the cylindrical walls. In this paper a similar disturbance consisting of alternating vortex filaments with their axes parallel to the flow direction is investigated. Besides the usual assumption of infinitesimal disturbance, the treatment is further simplified by assuming that the radius of curvature R of the wall is much larger than the thickness of the boundary layer δ and that the boundary layer profile remains unchanged along the wall. The dynamical equation is then reduced to a pair of simultaneous ordinary differential equations of second and fourth order. With a given boundary layer profile and the "Reynolds number" $(U_0\vartheta/\nu)(\vartheta/R)^{\frac{1}{2}}$ the six homogeneous boundary conditions reduce the problem to that of finding the characteristic values which are related to the damping coefficient and wave length of the disturbance; U_0 is the velocity outside the boundary layer, ν the kinematical viscosity and ϑ the "momentum thickness" of the boundary layer. This characteristic value problem is solved by transforming the differential equations into integral equations and then approximating the integrals by finite sums. Trial calculations show that it is necessary only to divide the boundary layer into four intervals in calculating the integrals, and that the influence of the shape of boundary layer profile is not large. In the final calculation, Blasius' boundary layer profile for flat plate is used. Curves of constant value of $\beta\vartheta^2/\nu$ are plotted using $(U_0\vartheta/\nu)(\vartheta/R)^{\frac{1}{2}}$ as ordinate and $2\pi\vartheta/\lambda$ as abscissa; β is the negative damping coefficient and λ is the wave length of disturbance. For large values of $2\pi\vartheta/\lambda$, an asymptotic solution is used. The lowest value of $(U_0\vartheta/\nu)(\vartheta/R)^{\frac{1}{2}}$ for instability to occur is 0.58 which corresponds to $\lambda \approx 45\vartheta \approx 5.0\delta$. Therefore, instability occurs only for $R > 0$, that is, concave walls. The rather uncertain measurements on transition from laminar boundary layer to turbulent boundary layer along a concave wall made by M. and F. Clauser [Nat. Adv. Com. Aeronautics, Tech. Note No. 613 (1937)] give $(U_0\vartheta/\nu)(\vartheta/R)^{\frac{1}{2}} \approx 9$.

The right of equation (6.1) should be multiplied by σ^2.

H. S. Tsien (Pasadena, Calif.).

Kibel, I. A. Vortex-motions of compressible fluids. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 8, 20–24 (1939). (Russian) [MF 3338]

The author considers the stationary two dimensional flow of an ideal compressible fluid under adiabatic conditions and with no exterior forces. The pressure p, the density ρ and the velocity v are connected by the relations

$$p^\gamma = \rho\theta, \quad \tfrac{1}{2}v^2 + (1-\gamma)^{-1}\theta p^{1-\gamma} = i_0,$$

where $\gamma = c_v/c_p$, $h(x, y) = h = \text{const.}$ is the equation of the stream lines and θ and i_0 are functions of h. In his previous paper [Leningrad State Univ. Annals 17, 79–95 (1937)] the author showed: Suppose i_0/θ is known as a function of h; to every $\chi(x, h)$ which satisfies a certain partial differential equation there corresponds a flow, the stream lines $y = y(x, h)$ of which can be determined by quadratures. The author investigates motions corresponding to certain special types of the function $\chi(x, h)$.

S. Bergmann.

Rosenblatt, Alfred. Sur la théorie mathématique de la lubrication des coussinets. C. R. Acad. Sci. Paris 210, 694–695 (1940). [MF 3916]

Cf. these Rev. 2, 25.

Burgers, J. M. On the application of statistical mechanics to the theory of turbulent fluid motion. A hypothesis which can serve as a basis for a statistical treatment of some mathematical model systems. I. Nederl. Akad. Wetensch., Proc. 43, 936–945 (1940). [MF 3723]

A discussion is given of the relation between P and U obtained in the paper reviewed below. An "effective length" N_e of the spectrum of elementary components of the secondary motion is defined by means of the equation $U = 6\nu N_e$. A general expression is adopted for the stream function of the secondary motion for the case of motion in a channel and the dissipation of energy per unit of length of the channel is calculated therefrom. The average statistical state

of the system is then derived on the assumption that the average of this dissipation per unit time has a given value.

H. Bateman (Pasadena, Calif.).

Burgers, J. M. On the application of statistical mechanics to the theory of turbulent fluid motion. A hypothesis which can serve as a basis for a statistical treatment of some mathematical model systems. II. Nederl. Akad. Wetensch., Proc. **43**, 1153–1159 (1940). [MF 3725]

A solution $\psi(y, t)$ of the basic equations for a restricted domain $0 \leqq y \leqq 1$ is supposed to be developable in a series $\sum \zeta_n \sin(n\pi y)$ in which the complex coefficient $\zeta_n = \xi_n + i\eta_n$ has an absolute value which cannot lie below a certain threshold value $\delta = \beta\nu$, where ν is the "coefficient of friction." It is assumed further that ξ_n and η_n may take only values of type $k\delta$, where k is an integer. Such a set of pairs of values of ξ_n and η_n determine a lattice point representing in a multidimensional space an instantaneous state of the secondary motion of the system described by the course of $\psi(y, t)$ at that instant. The number of times that the system is in the instantaneous state represented by the lattice point numbered m is denoted by f_m. Various statistical types of behavior of the system are distinguished by a suffix s and the corresponding numbers fulfill the relation $\sum_m f_{ms} = M$, where M is the total number of representative points in the phase space. In each instantaneous state m of the system the dissipation of energy per unit time by the secondary motion has a definite value

$$\epsilon_m = \tfrac{1}{2}\nu\pi^2 \sum_n n^2(\xi_{nm}^2 + \eta_{nm}^2).$$

Each sequence of instantaneous states must also fulfill the relation $\sum f_{ms}\epsilon_m = MPU$, where U is the velocity of the principal motion and P is the exterior force.

The average statistical state of the system is now specified by a set of numbers $\bar{f}_m$ calculated by Fowler's method with the aid of a "partition function" $F(\sigma) = \sum_m \sigma^{\epsilon_m}$, where σ is an auxiliary complex variable. The equation for $\bar{f}_m$ is indeed

$$\bar{f}_m \int_\gamma F^M d\sigma / \sigma^{1+PUM} = \int_\gamma M F^{M-1} d\sigma \cdot \sigma^{\epsilon_m - 1 - PUM},$$

where γ is a simple contour described in the counterclockwise direction around $\sigma = 0$. The ratio F^M/σ^{PUM} is infinite for $\sigma = 0$ and $\sigma = 1$; consequently it has a minimum value at a point σ determined by the equation $\sigma dF/d\sigma = PUF$, and the values of the integrands at this point play an important role when the integrals are estimated by the method of steepest descent. It is thus found that $\bar{f}_m/M = \sigma^{\epsilon_m}/F$. All kinds of average values pertaining to the system may now be calculated. The analysis for $\overline{\xi_n^2}$ and $\overline{\eta_n^2}$ involves the theta function $\vartheta_3(z, q)$ and includes an approximation which indicates that for small values of n there is equipartition of dissipation, $\overline{\xi_n^2}$ and $\overline{\eta_n^2}$ being inversely proportional to n^2, while for large values of n they decrease more rapidly as n increases. A calculation of the total dissipation indicates that U is approximately equal to $(21/\beta)\sqrt{P}$.

H. Bateman (Pasadena, Calif.).

Friedlander, F. G. The reflexion of sound pulses by convex parabolic reflectors. Proc. Cambridge Philos. Soc. **37**, 134–149 (1941). [MF 4240]

Consider a plane wave with arbitrary velocity potential $f(ct-x)$ incident on a convex paraboloid of revolution. Introduce parameters $a\xi = ct - x - 2a$, $a\eta = ct - (x^2+y^2+z^2)^{\frac{1}{2}}$; a is the focal length. Assuming the velocity potential ϕ depends on η and ξ only, the wave equation reduces to a hyperbolic equation in these variables, which can be integrated by two quadratures. The conditions on the general solution, namely, reflection at the surface of the paraboloid and $\phi(\eta, \xi) \to f(\xi)$ for $\eta \to -\infty$, yield the Volterra type integral equation

$$(1) \qquad l(\eta) + \textstyle\int_{-\infty}^{\eta} l(\nu)(\eta - \nu + 2)^{-2} d\nu = 2df/d\eta,$$

and ϕ is then determined from

$$(2) \qquad \phi(\xi, \eta) = f(\xi) + \textstyle\int_{-\infty}^{\eta} l(\nu)(\xi - \nu + 2)^{-1} d\nu.$$

The simplifications introduced by the choice of the parameters ξ, η and equations (1) and (2) constitute the essential novelty of this paper. The contributions of the incident and reflected wave are localized in the first and second terms on the right hand side of (2), respectively. A solution of (1) may be obtained, under suitable regularity conditions on f, by Fourier transforms [since the integral is manifestly a convolution] but the author prefers to use, for numerical computation, the first five terms in the series expansion of the resolvent kernel. Actually the author restricts himself to the case of a pulse, $f(\xi) = 0$ for $\xi < 0$, so that the integration range is now finite, that is, $0 \leqq \nu \leqq \eta$, and the developments converge. A similar analysis is given for the case of a parabolic cylinder with the difference that the exponents of the denominators in (1) and (2) are now 3/2 and 1/2, respectively.

D. G. Bourgin (Princeton, N. J.).

Rochester, Nathaniel. The propagation of sound in cylindrical tubes. J. Acoust. Soc. Amer. **12**, 511–513 (1941). [MF 4222]

The known theory of wave propagation in tubes with wall impedance determines the characteristic values as roots of a transcendental function and suggests the problem of the asymptotic expansions of these roots as the impedance $\to \infty$ or the reactance $\to 0$. For a tube with constant circular section the transcendentals are Bessel's functions and the author computes in a formal way what amounts to the first term in such asymptotic expressions by devices such as the replacement of the Bessel's functions by two terms of a Taylor's series. The results have some heuristic physical interest though the reader will have to correct the final formulae; for instance Eqs. (9) and (10) are false and (14) needs justification.

D. G. Bourgin (Princeton, N. J.).

Sacerdote, Gino. La densità di energia in alcuni problemi di acustica. Pont. Acad. Sci. Acta **3**, 47–52 (1939). [MF 4096]

Starting with the linear equations for the propagation of sound, the writer derives straight-forwardly the differential equations relating the scalar energy density E, the vector intensity $I = pu$ and u (p and u refer to pressure and velocity increment, respectively). If $u \operatorname{div} u = \frac{1}{2} \operatorname{grad} u^2$, the resulting equations involve E and I as the only dependent variables.

D. G. Bourgin (Princeton, N. J.).

Alvarez Lleras, Jorge. The fundamentals of tropical meteorology. Revista Acad. Colombiana Ci. Exact. Fis. Nat. **3**, 439–447 (1940). (Spanish) [MF 3702]

This is the first publication in a series in which the author proposes to give a physical explanation of the fundamental properties of tropical climate. The papers, written in the form of lectures, are based to a large extent upon the teaching of Garavito, the author's predecessor in the direction

of the National Astronomical Observatory at Bogota, and upon the author's own researches. The present publication deals mainly with Garavito's general representation of fluid dynamics and its application to questions of "the equilibrium of the atmosphere." *P. Nemenyi.*

Dederick, L. S. The mathematics of exterior ballistic computations. Amer. Math. Monthly 47, 628–634 (1940). [MF 3264]

The author gives a brief description of methods used in the mathematical theory of exterior ballistics. He indicates the method of Siacci for nearly horizontal trajectories and Kent's modification for nearly flat (but not horizontal) trajectories which are important for guns mounted in aircraft. He also mentions the methods based on numerical integration which were developed considerably by F. R. Moulton and Bliss between 1917 and 1925, and later extended by others. A suggestion is made as to improvements which are possible. *E. J. Moulton* (Evanston, Ill.).

Cioranescu, Nicolas. Sur les propriétés générales des mouvements balistiques. Acad. Roum. Bull. Sect. Sci. 22, 466–472 (1940). [MF 3203]

The author discusses the motion of the center of gravity of a projectile which is moving through a resisting medium in a plane. He is concerned with proving properties of the trajectory which hold regardless of the law of resistance, and establishes such facts as (1) the trajectory is concave downward, and (2) the horizontal component of velocity decreases throughout the motion. *E. J. Moulton.*

García, Godofredo. Mouvement des projectiles autour de son centre de gravité. Sur le mouvement gyroscopique; mouvement pendulaire des projectiles; dérivation. Revista Ci., Lima 42, 541–685 (1940). [MF 3143]

This lengthy paper begins with a resumé of previous physical assumptions such as Poisson's "gas cushion," the Magnus phenomenon, etc. Then five different sets of rectangular axes of reference are chosen and the relations between them and the different sets of Eulerian angles established. The external forces are classified as: (1) the resultant of the force of resistance and of the longitudinal friction, assumed to be in the plane of the axis of the projectile and the tangent to the trajectory; (2) the force normal to the above plane, due to surface friction; (3) forces opposing change of orientation of the axis. Expressions are obtained for these forces and their moments. (It is assumed that the trajectory and the velocity on the trajectory are known so that the principal force of resistance can be regarded as a known function of time.) The vector equation of linear momentum and the vector equation of moment of momentum are then expressed in various forms, according to the axes of reference chosen. On the basis of these equations the author discusses the various motions of the projectile about its center of gravity, treating oscillations, precession, nutation, etc., as well as the perturbing effect of these on the trajectory itself, giving rise in particular to the phenomenon of drift. Picard's method of successive approximations is used as a suitable method for solving the differential equations. In special cases the equations are reduced to linear form and transformed to the Volterra integral equation of the second kind. The paper is largely based on the work of Popoff. *W. E. Milne.*

Orbegoso, Guillermo. An application of the vectoral method of Dr. Godofredo Garcia to the solution of the principal problem of exterior ballistics. Revista Ci., Lima 42, 687–704 (1940). (Spanish) [MF 4344]

Thesis presented to obtain the degree of Bachelor of Science. Cf. the preceding review.

Sauer, R. Über Interpolation von Kurvenscharen mit Anwendung auf die Berechnung von Geschossflugbahnen. Z. Angew. Math. Mech. 20, 280–284 (1940).

Let's suppose that the coordinates x, y of a point of a simply connected region G are functions of two parameters t, λ; we suppose that the λ curves (that is, the curves $\lambda=\text{const.}$) and the t curves form a net. It is assumed that G is the set of points satisfying the inequalities $0\leqq t\leqq t_e$; $\lambda_a\leqq\lambda\leqq\lambda_b$ ($t_e, \lambda_a, \lambda_b=\text{const.}$; $\lambda_b>\lambda_a$; $t_e>0$). Furthermore it is supposed that every curve $\lambda=\text{const.}$ has at the point $t=0$ a tangent which is not perpendicular to the x-axis, and a curvature which is not equal to zero. The author finds a transformation T such that: (1) T carries the points (x, y) of G into the points (ξ, η) of a region Γ and carries the points of the line $t=0$ (on the boundary of G) into the origin $\xi=\eta=0$. (2) T carries the λ curves of G into curves (λ curves) of Γ, such that the correspondence between the points of a λ curve of G and the points of the corresponding λ curve of Γ is an affinity (the coefficients of which are functions of λ). (3) The values of $d\xi/dt$, $d^2\xi/dt^2$, $d\eta/dt$, $d^2\eta/dt^2$ for $t=0$ are independent of λ and $(d\eta/dt)_{t=0}=0$. The λ curves of Γ are tangent to the ξ-axis at the origin. If three curves $\lambda=\lambda_i$ ($i=1, 2, 3$) ($\lambda_i=\text{const.}$) are known, the determination of other λ curves requires interpolations. The author finds that in many cases it is simpler to perform the interpolations for the λ-curves of Γ, and afterwards to deduce the corresponding λ curves of G.

The main problem of the paper is the following. Suppose that we know some trajectories of a given projectile, corresponding to some values λ of the quadrant elevation (initial value of the inclination of the trajectory): $x=x(\lambda, t)$, $y=y(\lambda, t)$, where $t=\text{time}$. Suppose that the initial velocity is determined. We have to interpolate and to find other trajectories, corresponding to other values of λ. The author determines also the approximation which he can attain using the preceding method. *G. Fubini* (Princeton, N. J.).

Theory of Elasticity

Girkmann, K. Angriff von Einzellasten in der vollen Ebene und in der Halbebene. Ing.-Arch. 11, 415–424 (1940). [MF 4221]

The problems treated are those of a concentrated force acting in the plane of (I) an infinite plate and (II) a half infinite plate where the force vector is parallel or orthogonal to the boundary. The general Airy's functions F introduced are subsumed under

$$\Re(\mathfrak{Z})\int_0^\infty (A(\alpha)+yB(\alpha)) \exp \alpha(y\pm i(x-x_0))d\alpha.$$

The stresses are given by the three second partial derivatives of F. $A(\alpha)$ and $B(\alpha)$ are determined by the boundary stress conditions of the problem (including the vanishing of the stresses at ∞). The analysis is essentially formal. For

instance the concentrated load is written $\int_0^\infty \cos \alpha x d\alpha$. Integrals of this sort are freely combined and it is tacitly assumed that the equality of such "transforms" implies identity of the integrands. [Since the results obtained appear plausible, very likely the analysis can be recast satisfactorily in the usual way by introducing Cauchy principal values or limit operations.] *D. G. Bourgin.*

Schermann, D. I. Un plan élastique à coupures rectilignes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 627–630 (1940). [MF 3571]

A solution in terms of functions of a complex variable is given for the problem of plane stress in an infinite region which has a finite number of slits, all of them located on the same straight line. *E. Reissner* (Cambridge, Mass.).

Schermann, D. I. Problème mixte de la théorie du potentiel et de la théorie de l'élasticité pour un plan ayant un nombre fini de coupures rectilignes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 329–333 (1940). [MF 3248]

Let B be the z-plane with n slits along the real axis. The problem considered is the determination of an analytic function $\phi(z)$ regular in B, vanishing at infinity and such that on the upper edge of every slit $\Re(\phi)$, on the lower edge $\Im(\phi)$ are prescribed. By a formal calculation the problem is reduced to an integral equation which can be solved by the method of Carleman [Ark. Mat. Astr. Fys. **16** (1922)]. Thus the author obtains the solution, which consists of two terms of the form

$$\prod[(a_{2k}-z)/(a_{2k-1}-z)]\int_L \prod(a_{2k-1}-t)(a_{2k}-t)^{-\frac{1}{2}}g(t)(t-z)^{-1}dt.$$

Here a_r are the endpoints of the slits, L the totality of slits and $g(t)$ is a function related to the boundary values of $\phi(z)$ on L. The solutions obtained by the classical approach (for example, integral equations of Fredholm type) are less elegant. An analogous method is used for the determination of the plane state of stress when the exterior forces on the upper edges and the displacements on the lower edges are given. According to Mushelišvili the problem can be reduced to the determination of two analytic functions ϕ and ψ, such that

$$\delta(t)\phi(t)+\overline{t\phi'(t)}+\overline{\psi(t)}$$

is prescribed on the slits; $\delta(t)$ is a given function. By a procedure analogous to the above the author obtains explicit expressions for ϕ and ψ. *S. Bergmann.*

Scherman, D. I. Sur la solution du premier problème fondamental de la théorie de l'élasticité plane statique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 911–913 (1940). [MF 3586]

The author considers the problem of the determination of the displacements and stresses in a multiply-connected plane domain S, when the displacements on the boundary L of S are given. According to Kolosoff and Mushelišvili the problem above can be reduced to the determination of two analytic functions ϕ and ψ regular in S and satisfying on L the conditions

$$\chi\phi(t)-\overline{t\phi'(t)}-\overline{\psi(t)}=f(t).$$

Here χ is a constant and $f(t)$ is a known function. The author represents ϕ and ψ by integrals containing a new function $\omega(t)$, $t\epsilon L$. He then proves that $\omega(t)$ satisfies a nonhomogeneous integral equation of Fredholm's type. This equation is equivalent to a system of two real equations obtained by Lauricella [Nuovo Cimento (5) **13** (1907)]. The author proves that the homogeneous equation has only the solution $\omega(t)=0$. Therefore the nonhomogeneous equation always possesses a solution. *S. Bergmann.*

Scherman, D. J. Sur la solution du second problème fondamental de la théorie statique plane de l'élasticité. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 25–27 (1940). [MF 3594]

In this paper the author considers the same problem as in the paper reviewed above. However, he supposes now that the exterior forces are given on the boundary $L=\sum L_j$. The boundary conditions for the functions ϕ and ψ are

$$\phi(t)+\overline{t\phi'(t)}+\overline{\psi(t)}=f(t)+C_j$$

on L_j. The author again expresses ϕ and ψ by means of a function $\omega(t)$ defined on L and obtains a nonhomogeneous integral equation of Fredholm's type for $\omega(t)$. This equation differs from the corresponding equation established by Lauricella [Acta Math. **32** (1909)] for this case. The author proves that his equation has a unique solution.

S. Bergmann (Cambridge, Mass.).

Scherman, D. J. Problème mixte de la théorie statique de l'élasticité pour les domaines plans multiplement connexes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 28–31 (1940). [MF 3595]

Using the results of his previous notes [see the two reviews above] the author solves the mixed problem of elasticity in the two-dimensional case. Let S be a multiply-connected domain, the boundary of which consists of $2n$ arcs l_k. The exterior forces are given on l_{2k-1} and the displacements are given on l_{2k}, $k=1, 2, \cdots, n$. The functions ϕ and ψ [see above reviews] satisfy the boundary condition

$$\delta\phi(t)+\overline{t\phi'(t)}+\overline{\psi(t)}=f(t)+C$$

on L. Here $\delta=1$, $C=C_{2k-1}$ on l_{2k-1} and $\delta=-\chi$, $C=0$ on l_{2k}. The author again expresses ϕ and ψ with the aid of a function $\omega(t)$, $t\epsilon L$. Using the method of Carleman [Ark. Math. Astr. Fys. **16** (1922)] he obtains for ω a system of integral equations of Fredholm's type. The proof of the existence of the solution of the integral equations can be obtained in the same way as in the author's previous notes.

S. Bergmann (Cambridge, Mass.).

Gurevitch, C. G. Stability of two-dimensional stressed state. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **8**, 137–152 (1939). (Russian) [MF 3343]

Starting from a variational formulation of the stability problem of the elastic plate the author gives a derivation of the equation for the deflection of thin plates [see also H. Reissner, Z. Angew. Math. Mech. **5**, 475 (1925)]. Then he shows that the solution of the equation can be reduced to the solution of a linear integral equation with unsymmetric kernel. Applying the method of E. Schmidt he symmetrizes this kernel. Writing the Green function in the form $\sum_n w_n(x, y)w_n(\xi, \eta)/\mu_n$, he obtains an infinite system of linear equations for the characteristic values and functions. Considering only a finite section of this system, one obtains a secular equation for the approximate determination of the characteristic values. In the case of a simply supported rectangular plate, one obtains $w_n(x, y)=(\sin m\pi x/a)(\sin n\pi y/b)$, where a and b are the length and the width of the rectangle. The same approximate solutions were obtained by Timo-

shenko, S. Bergmann and H. Reissner [Z. Flugtech. Motorluftschiffahrt **23**, 6 (1932)] by the method of Ritz and using variational methods. *S. Bergmann.*

Savin, S. A. Method of integral algebraical functions in the theory of elasticity of three dimensions. J. Math. Phys. Mass. Inst. Tech. **20**, 1–17 (1941). [MF 3839]

The author presents a further discussion of the use of his method of expressing the components of displacement by polynomials [J. Math. Phys. Mass. Inst. Tech. **17**, 201 (1939)]. He is concerned here with the relations between the coefficients in the expressions for the components of the stress and those in the expressions for the displacement and with the determination of the coefficients in problems to which the method is applicable with the aid of the equations of equilibrium and the boundary conditions.
H. W. March (Madison, Wis.).

Savin, S. A. Saint Venant's conditions of compatibility in the method of integral algebraical functions. J. Math. Phys. Mass. Inst. Tech. **20**, 18–22 (1941). [MF 3840]

Relations among the coefficients in the expressions for the stress components [see the preceding review] are obtained from St. Venant's condition of compatibility of strain components. *H. W. March* (Madison, Wis.).

Savin, S. A. On some solutions of the equations of internal equilibrium of the theory of elasticity. J. Math. Phys. Mass. Inst. Tech. **20**, 23–29 (1941). [MF 3841]

A discussion and comparison of methods of solving the equations of equilibrium of an elastic solid.
H. W. March (Madison, Wis.).

Ôkubo, Hajimu. The stress distribution in a semi-infinite domain having a plane boundary and compressed by a rigid body. Z. Angew. Math. Mech. **20**, 271–276 (1940). See these Rev. **1**, 287.

Talypov, G. B. Stability of a compressed infinite strip in a resistant medium. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **8**, 128–136 (1939). (Russian) [MF 3342]

Using the method of Southwell [Philos. Trans. Roy. Soc. London. Ser. A. **213**, 187 (1913)] the author obtains in the usual way the equations for the displacements of an infinitely thin plate embedded in a resistant medium. [See also Biezeno and Hencky, Nederl. Akad. Wetensch., Proc. **31** (1928); Biot, Proc. Fifth Intern. Congress Appl. Mech., 1938, p. 117, and others.] Furthermore, he calculates some particular solutions and determines the critical pressure corresponding to these solutions. *S. Bergmann.*

Despujols, Pierre. Sur les réactions élastiques d'une dalle soumise à des forces cylindriques. C. R. Acad. Sci. Paris **210**, 659–661 (1940). [MF 3050]

In a previous paper [C. R. Acad. Sci. Paris **210**, 206–208 (1940); these Rev. **2**, 32] the author has studied the displacements and elastic forces in a beam of infinite thickness subject to an external force. Here the author uses previous results to study an indefinite slab of finite thickness subject to exterior forces in static equilibrium. As before, physical considerations are used to reduce the problem to one in two dimensions. *E. W. Titt* (Hyattsville, Md.).

Baron, Francis M. Influence surfaces for stresses in slabs. J. Appl. Mech. **8**, A-3–A-13 (1941). [MF 3894]

Bending moments and twisting moments at points in a slab, associated with a concentrated load, are calculated using Nadai's stress function. In a number of examples contour lines are shown of the surface representing the moments at a given point due to a concentrated load at an arbitrary point. *H. W. March* (Madison, Wis.).

Federhofer, K. Berichtigung zu meinem Aufsatze in Band XI, S. 224 des Ingenieur-Archivs "Knickung der Kreisplatte und Kreisringplatte mit veranderlicher Dicke." Ing.-Arch. **11**, 386 (1940). [MF 4220]

Cf. these Rev. **2**, 174.

Sakadi, Zyuro. On elasticity problems when the second order terms of the strain are taken into account. Proc. Phys.-Math. Soc. Japan (3) **22**, 999–1009 (1940). [MF 3797]

The author considers an extension of the classical theory by introducing formal expressions, which are identical in structure with strains and rotations of the classical theory, and by utilizing the stress-strain relations which involve four new elastic constants in addition to the usual constants λ and μ. The meaning of the new elastic constants is not revealed. *I. S. Sokolnikoff* (Madison, Wis.).

Pizzetti, Giulio. Sul problema dell'equilibrio elasto-plastico dei tubi. Pont. Acad. Sci. Comment. **4**, 147–168 (1940). [MF 4110]

Green, A. E. Note on general bi-harmonic analysis for a plate containing circular holes. Proc. Cambridge Philos. Soc. **37**, 29–33 (1941). [MF 3768]

The formulae of a previous paper [Proc. Roy. Soc. London. Ser. A. **176**, 121–139 (1940); these Rev. **2**, 31] are extended so as to include stress distributions with force resultants on the edge of the holes. *E. Reissner.*

Novojilov, V. Computation of tensions in a thin spherical shell in the case of an arbitrary load. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 537–540 (1940). [MF 3225]

The equations for the elastic deformation of a thin spherical shell are treated in the following manner. Let θ and ϕ be defined as spherical coordinates, u, v and w as displacement components,

$$U=\frac{\partial u}{\partial\theta}+u\cot\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}-(1+\nu)w,$$

$$V=\frac{\partial v}{\partial\theta}+v\cot\theta-\frac{1}{\sin\theta}\frac{\partial u}{\partial\phi}$$

as auxiliary functions and

$$\Delta\equiv\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

as the Laplace operator in spherical coordinates. The equations of the problem may then be reduced to

$$\Delta\Delta w+2\Delta w+c^2w=f_1,\quad \Delta V+2V=f_2,\quad \Delta U+2U=f_3,$$

where $c^2=12(1-\nu^2)a^2/t^2$ and the f's are certain functions of the external surface loads. The integration of this system is effected in terms of spherical harmonics. *E. Reissner.*

Reissner, Eric. A new derivation of the equations for the deformation of elastic shells. Amer. J. Math. 63, 177–184 (1941). [MF 3641]

The author derives the classical equations of the linear theory of bending of thin shells in a clear and compact way. This is achieved through the use of a special coordinate system. The points in the shell are located by a vector

$$R(\xi_1, \xi_2, \zeta)=r(\xi_1, \xi_2)+\zeta n(\xi_1, \xi_2),$$

where r is a vector to the middle surface of the shell, n a unit vector normal to the middle surface and $\xi_1=$const., $\xi_2=$const. are the lines of curvature of the middle surface. The equations of equilibrium and the stress-strain relations are then derived with surprising ease and brevity, especially by comparison with the derivations in the standard treatises.
J. J. Stoker (New York, N. Y.).

Lee, E. H. The impact of a mass striking a beam. J. Appl. Mech. 7, A-129–A-138 (1940). [MF 3258]

An approximate solution of this problem is given, on the basis of H. Hertz's theory of elastic impact, by equating the gain in kinetic and strain energy of the beam during the time of impact to the corresponding energy loss of the striking mass, with an assumed plausible law of contact force variation. *E. Reissner* (Cambridge, Mass.).

Wigglesworth, L. A. Flexure and torsion of a circular shaft with two cracks. Proc. London Math. Soc. (2) 47, 20–37 (1940). [MF 3649]

The solution of St. Venant's torsion and flexure problems for a circular shaft with two slits extending from the ends of a diameter to any depth along the diameter is obtained in closed form. The problems are solved by mapping the area of the cross-section of the shaft conformally on a half-plane and applying the formula of Schwarz. The paper contains numerical calculations for some particular cases and a comparison of the author's results with those obtained by Shepherd [Proc. Roy. Soc. London (A) 138, 607–634 (1932); 154, 500–509 (1936)]. It appears likely from this paper that the infinite series obtained by Shepherd for the flexure function can be summed. *I. S. Sokolnikoff.*

Carrizosa Valenzuela, Julio. Deduction of the equation of elasticity of Kriso and Baes for the calculation of the beam of Vierendeel by means of the relations of deformations of Bresse. Revista Acad. Colombiana Ci. Exact. Fis. Nat. 3, 397–405 (1940). (Spanish) [MF 3701]

The Vierendeel beam refers to a class of rigid frameworks with no diagonal bracing. The known stress analysis reduces to consideration of a unit consisting of a member and the neighbor on each side of it. The author develops such an analysis, starting with the virtual angular (and linear) deviations in the form

$$\Delta\phi=-\int_s^{s+\Delta s}\frac{M}{EI}ds, \tag{1}$$

where the notation is standard. (The classic formula (1) is ascribed by the author to Bresse.) *D. G. Bourgin.*

Nicolai, E. L. Euler's works on the theory of struts. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 8, 5–19 (1939). (Russian. English summary) [MF 3337]

In this lecture, delivered at the Research Institute of Mathematics and Mechanics (University of Leningrad), a short account of the works of Euler on the theory of struts is given. It is pointed out that several results, obtained in recent times, are really due to Euler. An approximate formula of Mises for the deflexion of a compressed bar beyond the limit of stability is to be regarded as included in a more general result of Euler. The problem of Greenhill on the stability of a vertical rod under its own weight was originally solved by Euler. *Author's summary.*

Krall, Giulio. Problemi della dinamica dei ponti. Pont. Acad. Sci. Acta 3, 5–25 (1939). [MF 4094]

Limits are obtained for the vibrations of a beam supported at both ends and carrying a load concentrated at a single point which moves with uniform velocity from one end of the beam to the other. The displacement at time t of a point at a distance x from one end is represented by $w(x, t)$, which is developed in a series of the form $\sum_{k=1}^{\infty}\varphi_k(t)u_k(x)$, where the u's are a suitably chosen set of orthogonal functions. In the important case of a homogeneous beam of length l, $u_k(x)=$const. sin $(k\pi x/l)$. The results are obtained from an analysis of the infinite systems of differential and integro-differential equations satisfied by the $\varphi(t)$'s. The precise form of these equations depends on the nature of the load. If, for instance, the load is without inertia, the equations are especially simple, the variables are separable, and their solutions can be written down explicitly in the form

$$\varphi_k(t)=(1/\sigma_k)\int_0^t F_k(\tau)\sin\sigma_k(t-\tau)d\tau,$$

where the F's are known functions and the σ's are known constants. *D. C. Lewis* (Durham, N. H.).

Lindsay, R. B. Elastic wave filtration in nonhomogeneous media. J. Acoust. Soc. Amer. 12, 378–382 (1941). [MF 3803]

The author is concerned in part with establishing that for sufficiently "smooth" variations in medium properties a plane sound wave incident normally on a transition layer is propagated without attenuation. Some other material of physical interest is also presented. The mathematical equivalent amounts to a study of $d^2p/dx^2+k^2(x)p=0$ subject to boundary conditions. A first approximation only is taken to the solution and the treatment is patterned on Rayleigh [Proc. Roy. Soc. London. Ser. A. 86, 207 (1912)]. The author's proof is not quite complete, for his assumption of the adequacy of the first approximation involves the existence of a non-negative variable function in C^1, vanishing at two points of a finite interval, for which the first derivative is everywhere negligible in comparison with the function. The theorem of the mean shows these conditions cannot be satisfied. Further slight extension of the argument, involving either a higher approximation or a bound for the influence of the error made in taking the first approximation, seems required. *D. G. Bourgin.*

AUTHOR INDEX

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